

immediate educational assistance

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SIMPLE MATH

For Children and Adults

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Simple Math for Children and Adults. —

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Many parents fall into despair after yet another unsuccessful attempt to help their child with homework. If the child is still in elementary school, one attempt follows another. But when the child is already in 8th or 10th grade—give up hope... All help boils down to a shout: “Have you done your homework?!”

What should you do? Look for *a Teacher*. And while you’re looking for one, some quick educational help will come in very handy.

An original methodology, named by the author *the Method of Basic Abstractions*, makes it easy to master the basic math curriculum—from counting on sticks and natural numbers to trigonometric equations and the study of functions using derivatives. The material is, for the most part, presented in the form of a dialogue between parents or a teacher and a child.

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1. Foreword

Never help your child with their schoolwork. You don't know how to do it, and you'll only ruin everything.

First, all your "help" will amount to you doing the homework instead of your child. And that teaches them to cheat. After all, at school, the child will pretend that they did the homework themselves.

Second, by checking the assignments, the teacher gathers information on which to base their conclusions. Your interference distorts this information.

Doing homework means practicing what has already been learned. Therefore, it is better if, instead of you helping your children, they help you understand the new things they have learned and discovered at school.

And, of course, never scold your child. Why? Just don't scold them—that's all!

They'll soon get used to being scolded all the time and stop paying attention. What's more, constant scolding will make them convinced that you don't love them and that you don't need them. It will be hard to change their mind. Even if you stop scolding them later and are kind, they'll still look at you with distrust for a long time. Very often, because of this, parents end up yelling again: "Well, what more do you want?! We've done this and that for you, and you're still giving us the evil eye!" In doing so, they destroy the child's last hope that you were sincere.

A ban on scolding does not mean a ban on punishment. Just don't confuse punishment with retribution, vengeance, or violence.

Punishment is one element of learning—learning to act according to the rules. It comes from the word “na-kaz” (command). And you must be prepared to punish a misdeed. But you must clearly explain why the child is being punished and exactly which rule they broke. Moreover, the child must be aware of this rule in advance, and they must also be aware of the punishment for breaking it.

The punishment can be quite simple: you can send them to the corner, you can forbid them from leaving their room, you can take away their right to watch TV or go out into the yard. But you must say this while looking them straight in the eye, and, most importantly, you must clearly specify the duration of the punishment. If the duration is specified, then under no circumstances should it be violated. Your “pity” will be interpreted either as weakness or as your own violation of the rules of the game. Therefore, before announcing the punishment and its *duration*, think it through carefully.

And remember, it is not the severity of the punishment or its length that teaches a lesson, but the consistency and firmness with which it is enforced. Let the child sit in a dark room for exactly two minutes as punishment for the misbehavior. The main thing is that they know why they are being punished and that after two minutes have passed, you will accept them again, and no one will ever mention their misbehavior again. You must not punish (even with a reminder) the same misbehavior twice.

Play with your children more often. But remember: play is a competition governed by rules that everyone must follow—you, first and foremost.

The first rule of any game is: you can't force anyone to play! It's best if you reluctantly agree to the children's suggestion to play.

The second rule of the game is that the duration of the game must be clearly established in advance. You must set a specific time for when the game should end. Under no circumstances should you agree to continue the game beyond the predetermined time. Children, especially when excited by the game, lose track of time and can wear themselves out completely without even realizing it. You need to help them wrap up the game.

But where can you find games, especially ones with an educational focus?

They're all around you. You just need to learn to see them—or... remember a little. After all, you were once a child too.

2. How do you ask questions?

The most interesting game is one that springs up right here and now, as if by magic. Become magicians for your children. It's not hard. It's not hard as long as they believe in you.

The easiest way is to pretend that you don't know something that seems like the simplest thing you should know. This approach is the exact opposite of the completely unacceptable situation where an angry dad yells at his child: "What?! You don't know such simple things?!!"

There are no simple things for children.

For example, ask your child if they measured anything in class. It doesn't matter what. Then ask them, "What does it mean *to measure*?" Pretend you don't know what that means. I think many of you won't even have to pretend. But that's wonderful, because it will be honest, and you'll give your child a chance to be the teacher. The main thing is to listen honestly, speak honestly about what you understood and what you didn't, always agree with valid points, and never try to blurt out the correct answer before the child does. And don't forget to sincerely thank them for the lesson if it goes well. Because a real lesson is always a two-way street: both the teacher and the student learn from it. A teacher needs a student just as much as a student needs a teacher.

The teacher's main tool is the question. "To teach is to ask wisely." So said the great teacher Alcuin.

We, too, will try to ask, if not wise, then at least simple questions, with the specific aim of teaching.

For example:

- What does it mean to measure?
- Well, it means to measure the length...
- But I don't know what it means to measure.
- Well, to measure with a ruler...
- And what do you have to do with a ruler to, say, measure the length of a table?
- You have to do it like this...
- Don't show me, just tell me. I'll do whatever you say. Give me the ruler. So, what do you say I need to do with the ruler to measure?
- You have to put the ruler on the table...
- But it won't stay there...
- Well, put it down...
- I put it there!
- Not there...
- Where then?
- On the corner of ...
- The whole ruler? Right on the corner ? Please...
- No, not like that!
- How then?
- Place it at the beginning of the table...
- And where is the beginning of the table?
- Well, at the edge of the table...
- Here, the ruler is at the edge of the table...
- Not like that!
- How then?
- ...
- etc.

This is one example of how something that seems simple can suddenly become very complicated. Yet all it took was simply defining the dimension .

But what is *a definition* ?

In textbooks, definitions are highlighted in bold, but practically nowhere is it explained what they actually are. And you yourself have probably never asked yourself what a “definition ” really is.

So, your task is to make everything complicated at first. After all, if you succeed, it means the child’s knowledge wasn’t genuine. They simply memorized the terms and used them. But did they actually understand them?

Just like in medicine: first comes the diagnosis, identifying the problem, understanding the problem, and only then the treatment. But if you can’t find the problem, that doesn’t mean it doesn’t exist. It may turn out that you simply can’t see it. This is where professionals should step in.

And in this book, dear reader, problems will appear where you least expect to find them.

3. What is a definition?

Definitions in textbooks are often highlighted in bold, and teachers constantly insist that we memorize them... But what is a definition, and how is it constructed?

To start, let's take something straightforward and gradually move toward the question at hand. Let's take, for example, measurement.

Measurements accompany us throughout our lives. We're constantly measuring something. But describing the measurement process in words isn't something that comes right away. True, there's usually no real need for it. But a child must learn to describe in their own words what they're doing. This is very important! After all, otherwise, they won't learn to describe what they feel.

To start with, let them use your help. Let's try to describe the procedure for measuring the length of a table.

Example: "I take a ruler. I place it on the table. I move it so that one end of the ruler aligns with the near left corner of the table. Then I place the ruler along the side of the table whose length I want to measure. I mark the position of the ruler's second end on the table. Then I move the ruler so that the first end of the ruler aligns with my mark. I mark the new position of the ruler's second end on the table. And so on, until I reach the other corner of the table. Here, I make a mark on the ruler at the point that aligns with the corner of the table. Now I count the total number of centimeters. The length of the table is measured: I took the measurement " ."

This is what a description of how to measure the length of a table might look like. At first glance, it seems simple. But try asking your child to write it down, or try writing it out yourself. It'll be fun to see where the difficulties pop up in the most unexpected places. Turn those difficulties into fun challenges.

* * *

Every lesson should have a goal.

What was the goal when you asked what it means to measure? There had to be a goal. Without it, there was no point in asking. If the child isn't convinced that your question isn't idle curiosity but rather a lesson—that is, an action intended to teach—they'll stop noticing your questions. We teach a child to measure so that in the future they can describe the size of something in a way that another person will understand.

But is that the only reason they need to learn?

If that were the only reason, it is unlikely that people in ancient times would have ever started learning... A child begins to learn not to describe size and not because they are worried about their future. A child derives physical pleasure from the process of learning. This is an innate trait of theirs.

Once the child has learned to describe the measurement procedure without hesitation, ask them to come up with the shortest description. After a few unsuccessful attempts, help them by suggesting that the whole procedure is a comparison. Then intrigue them by saying that the shortest description will consist of just three words.

That is precisely our goal: to arrive at the most concise, briefest description possible.

*The shortest description is called **a definition** .*

At the initial stage, it is very important to teach the child to formulate definitions on their own.

After a few attempts, something like this might come out: to measure means to compare with a ruler... But what is a ruler? Let the child try to describe to you what a ruler is, and you pretend that you don't know what it is.

For example:

- What is a ruler?
- Well, it's like... a ruler...
- A ruler is a ruler?
- It's... it's like... a wooden... thing with markings on it, centimeters!
- And what are centimeters?
- Centimeters are what you use to measure length.
- Do we measure with a ruler or a centimeter?
- With a ruler...
- And what's a ruler?
- ?
- Can you measure length with a pencil?
- No...
- Well, why not? Here, I'll take a pencil and compare the length of the table to the length of the pencil: I got... 12 pencils.
- But pencils come in different sizes!
- Great! What about rulers?
- Same thing.
- Very good! What about rulers?
- But all rulers are the same!

To describe the length of something means to compare it to something whose length is known to everyone, to what is accepted as a unit of length. For us, that is one meter. Everyone knows it. And what is accepted as a unit of comparison and known to everyone is called *a standard*. A centimeter is one-hundredth of that standard.

Therefore, we can formulate a final definition :
measurement — comparison with a standard.

Now we can generalize. Let the child first try to briefly summarize the essence of everything that has been covered. You should only assist them.

It should be noted here that the “child” with whom lessons can be conducted in this way can be either 8 or 18 years old. The results will be the same.

4. What are terms?

Repetition is the mother of learning.

True. But repetition is needed not only for learning. Repetition helps the child notice their own successes. Success inspires and gives strength to move forward. Repetition reminds a child of who they used to be, helping them compare their current self with their past self and see the difference, noticing their own growth. Often, after such a comparison, they find themselves smiling at their former mistakes and fears. And this helps them cope with new fears.

A child doesn't notice how they're growing. And parents often don't notice it either. But once you start marking the child's growth with a notch on the doorframe, everything changes. You now have something to compare it to.

Let's review the content of the previous lessons.

Let the child do this first. Let him be the teacher now. You are just there to help. He should come up with something like this:

“To measure means to compare with a standard. A standard is a unit of comparison known to everyone. The unit of comparison for length is 1 *meter*. To measure the length of something means to compare it with 1 meter or 1 centimeter. The comparison is made using a ruler.”

Now it remains to note that to measure mass means to compare it with the standard of mass, i.e., 1 *kg*; to measure area means to compare it with the standard of area, i.e., 1 *square meter*, and so on. We have arrived at the universality of this definition.

It seems like that's all... Everything for this topic titled "Measurement"

Here comes the critical moment. The intrigue has vanished. The game is over, and a sense of disappointment sets in.

"Is that all?" a child seems to ask. "Will it just be the usual routine of ordinary lessons from here on out?"

No, of course not. The most interesting part is just beginning. And by the same rules as before. We will find the unusual and incomprehensible in what seems to be the most ordinary and understandable. We need paradoxical questions.

How can we learn to ask questions? After all, questions are the most important part of learning. Let us recall Alcuin: "To teach is to ask wisely."

The previous lesson ended with *us defining " " as the term " " and "measurement" as " ."* It seems like everything is clear. Well, go ahead and ask your child if everything is clear to them. They will, of course, answer that it is. Warn them that there's a trap waiting here. In reality, not everything is clear to them. But they will insist and even try to repeat the definition...

That's when you'll catch them off guard: what is a definition ?

They don't know that. I think you don't remember either. Although in the last lesson we gave a definition of " . " And now you're caught off guard too. What's more, the child might now go on the offensive and ask, "What is *the term ' ' ?*"

An ordinary person, unfamiliar with the previous lessons, will understand only one word from the expression "give a definition of the term the dimension "—namely, "give." He uses the other words more out of hab-

it. True, he doesn't need anything more; he finished his studies long ago. But those who are still studying and teaching need to have a clear understanding of every word. Otherwise, it will be impossible to study and teach.

So what is *the definition* of ? Let's look at it again. Let's follow the same procedure we used to answer the question, "What does it mean to measure?" Let's start with a description.

Draw a circle and ask the child to describe in their own words what they see. They will answer that it is a circle.

Now ask them not to use the word "circle" or any words related to it (circle, circle , etc.). They will most likely say it's a wheel or the sun or something else. Remind them that they should describe only what they see. There is neither a wheel nor the sun here. What is it?

The confusion the child will feel shouldn't upset or anger you. After all, you're a bit confused yourself. If you stop reading the text and try to describe what you've drawn on your own, you won't get it right away either. Put yourself in the child's shoes and work through the difficulties together.

After a few of their unsuccessful attempts, draw a straight line next to the circle . Ask, "What is this?" The child will say it's a line. "What kind?" "A straight line."

Now draw a curved line next to the straight line and ask what kind of line it is. Answers may vary. Children usually say it's a wavy or uneven line.

Remind them that all non-straight lines are called *curves*. Therefore, this is a curved line.

Now let's go back to the first picture. This is also a curve. How are the two curves different? Help your child see that one curve is closed, while the other is not.

So, what is this? It's a curved line drawn on a sheet of paper, and it's closed. Great!

Now close your second curve. Just draw a line with a pencil connecting the ends of the curve. Now it's closed, too. How do the two closed curves differ?

Place a point at the center of the circle and a point roughly at the center of your second closed curve. Yes, the first curve is equidistant from its point, while the second is not.

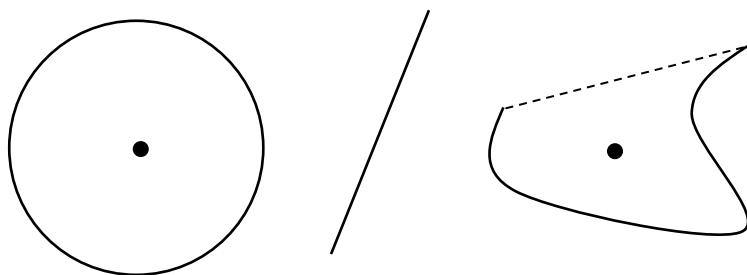


Рис. 1

The shortest possible description of what we've drawn is: *a closed curve equidistant from a given point.*

No matter who describes it, no matter who tries to find the shortest description, it will be the same for everyone, because the shortest version **is** always **unique**.

*The shortest description is called **the definition of a***

Since the definition is unique, it can be designated by a word and used exclusively from then on.

*Words that denote definitions are called **terms**.*

The term that we will assign to our derived definition—i.e., a closed curve equidistant from a given point—will be a *circle*.

You can see a spark in the child's eyes. He hasn't just understood something; he now knows how to do it. This is very important to him. And he's afraid of losing the knowledge he's just gained... Usually, children rush to write down what they've understood.

But no. We won't write anything down yet. We'll only write things down after we've learned to speak freely. So let's continue.

Let's try to describe something else using the same method. We have a circle and a given point. Ask the child to describe this point. Where and how is it located relative to the circle? It is equidistant from the circle. And how can we say that in just four words? The child already knows: a point equidistant from the circle! We've got a definition. Let's give it a name. It will be *the center of the circle*.

The center of a circle is a point equidistant from the circle.

Now let's connect two points on the circle with a line segment. (Рис. 2).

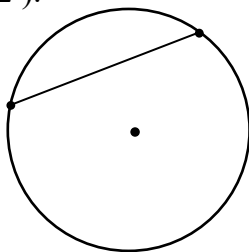


Рис. 2

Ask your child, "What is this?" Have them describe it and come up with the shortest description (*definition*:).

Hint that it is a line segment. But not just any line segment—one whose endpoints lie on the circle. Tell them that the shortest version should consist of only five words.

The result should be the definition : *a line segment connecting two points on a circle.*

We'll immediately tell you the word used to describe this definition—**chord** . This is *a new term* .

A chord is a line segment connecting two points on a circle.

Construct a line segment connecting two points on a circle so that it passes through the center of the circle (рис. 3). Ask: Is this a chord ?

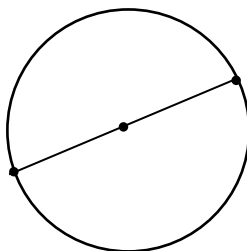


Рис. 3

In most cases, children say no.

Ask them to describe it. Get the child to say that it is a line segment connecting two points on the circle. And what is the term for it? Of course, a chord . But unlike other chords, this chord passes through the center of the circle. It divides the circle into two equal parts and therefore deserves a separate term —*a diameter* .

A diameter is a chord that passes through the center of the circle.

Note that to formulate this definition, we used terms introduced earlier: “chord” and “circle” .

Let's connect a point on the circle to its center (рис. 4). We've obtained *a line segment connecting a point on the circle to its center*. This is ***the radius*** .

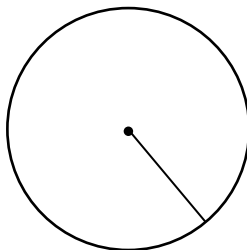


Рис. 4

So, everything works! We have learned how to construct definitions and assign terms.

Now we know what we sort of knew before.

Tip for parents: Before teaching this lesson to your child, practice it with each other first. :))

And how does a term differ from a name?

5. How is a term different from a name?

Children are already familiar with names. When introducing children to a new object, we show it and say: this is called such-and-such. That's how we teach names. How does a name differ from a term? Can a name be a term?

Terms are assigned to definitions. Names are assigned to objects. An object has no definition. We cannot construct the shortest description of a real apple or pencil. We cannot even construct a standard description. After all, it will never be complete. Any object has an infinite number of physical characteristics. For example, take that same pencil: there's its length, the number of sides, its weight, its color, the composition of the lead, how sharp it is... It's impossible to describe everything. That's why we show the pencil as a whole and say: this is called a "pencil."

"Pencil" is a name, i.e., a noun—a word that designates this real object. So, a name is a noun. A noun is a word that designates an object.

But isn't a circle a noun? No. We cannot directly see a circle. It is an abstract object. What is drawn on the board in the form of a circle is just a line, a mark left by chalk. And a circle is a closed curved line... A line is a set of points. And a mathematical point has no dimensions—it is an abstraction. A point drawn with chalk on a blackboard or with a pen in a notebook is not a point. It is a spot that we mistakenly call a point. A spot can be seen; it has dimensions. A point cannot be seen, which means it cannot be described. Therefore, a point has no definition.

They say a point is a limit term. The point itself cannot be described. But with the help of a point, everything else can be described.

For example, we say that a straight line is a set of points arranged side by side... But there is a contradiction here. Since a point has no dimensions, a set of points “side by side” is also a point. That is, a straight line cannot be seen or described. Since a point is nothing, a straight line is also nothing. A straight line is an abstraction. A straight line drawn with chalk on a board is a mark left by the chalk. A mark is a noun. But a straight line is not a noun; that is, it is not a word denoting an object. There is no object called a straight line. It is an abstraction. Words denoting abstract objects are called *terms*.

If a person manages to understand, imagine, or observe an abstract object, then it begins to exist for them. That is, for some people, a circle is a term, while for others, it is a noun. Our task in working with children is to ensure that terms become nouns for them. And to do this, we must help them answer the question “What is this?” regarding an abstract object that cannot be directly demonstrated, like a pencil. A child must learn to use terms just as they use names.

But how do we describe something that does not exist? This is where the method of basic abstractions comes into play. We take a basic term and use it to describe higher-order terms. Using a point, we describe a line; using a line, we describe a plane; using a plane, we describe space. None of these terms has a definition, but only a measure of description. A point has no measure—it is zero-dimensional (having no measure is also a measure). A line is described by a point—it is one-dimensional. A

plane is described by a point and a line—it is two-dimensional. Space is described by a point, a line, and a plane—it is three-dimensional.

These four fundamental terms allow us to construct descriptions and definitions of any abstract objects. Since every abstract object is nothing, it can be described by a finite number of fundamental terms of its constituents. After all, there are only four of them: a point, a line, a plane, and space. Or even just a single fundamental term—a point.

A real object cannot be described by a finite number of names of its constituent elements. After all, their number, down to the atoms, is infinite.

Thus, it is impossible to fully describe any real object, but it can be directly demonstrated and designated by a word. This word is a noun, a name.

An abstract object cannot be demonstrated. It does not exist. But it can be described using limit terms. The shortest description—a definition—can be designated by a word. This word (term) is already a noun.

We have demonstrated how to turn an abstract object, which has only a term, into a noun. But can we demonstrate the reverse process? Can a noun become a term and, through that, an abstraction?

It can.

For a child, at first the name “apple” refers only to this specific red apple. If you show them another apple, say a green one, and they compare it to the first and find a difference, they won’t accept it as an apple. To them, it isn’t an apple. It’s something else. Only after some time, through social interaction, does the child understand that there are many objects which, no matter how different

they may seem, are all still called apples. The child forms a general image of an apple (an abstraction), against which they distinguish the observed object as an apple, even if they have never seen an apple of that kind before. In this case, for the child, the word “apple” is no longer a name but a term.

Thus, we have traced two opposing movements: from abstraction to reality and from reality to abstraction. The ability to move freely in both directions is precisely what we must teach our child. This is the essence of Education, that is, the development of the ability to translate reality into images (abstractions) and, conversely, images into abstractions.

6. How do fairy tales help teach lessons?

Fairy tales, especially for the very youngest, are always lessons. Every fairy tale is a lesson. What kind?

Ask your child to tell the shortest fairy tale for the very youngest. And specifically, a fairy tale, not a magical story written by some author.

And how do you tell a real fairy tale apart?

In a real fairy tale, every word is always in its right place. That's *the definition*—the shortest possible description.

So, the child is thinking. They just can't decide what to tell. Ask them to tell, for example, "The Turnip."

The ways the fairy tale might begin can vary. It could be "Once upon a time, there lived an old man and an old woman...", or "Once upon a time, an old man planted...", or "Once upon a time, there was a turnip..."

The beginning must be only: "The old man planted a turnip." This is very important.

Next, of course, it should go: "The turnip grew big, bigger than big. The old man went to pull the turnip."

Not "he began to pull" or "he started to pull it out," but specifically "the old man went to pull the turnip."

"He pulls and pulls, but he can't pull it out! Grandpa called Grandma. Grandma grabbed Grandpa, and Grandpa grabbed the turnip..."

The order must not be broken. Tell the story a hundred times, and a hundred times it must be the same. Every word in its place.

"They pull and pull, but they can't pull it out. Grandma called her granddaughter!"

Why is this important? A toddler, hearing a fairy tale for the first time, may not understand anything at all. They're drawn only to Mom's voice and intonation. But time and again, listening over and over, they begin to distinguish individual words and phrases. But that's only if the word order hasn't changed each time. Swap two words, and the child will think it's a different fairy tale.

"The granddaughter pulls for Grandma, Grandma pulls for Grandpa, and Grandpa pulls for the turnip. They pull and pull, but they can't pull it out."

Things will move faster from here on out. Soon the story will be told. Now we can analyze it. What lesson is contained in this story?

The child will immediately guess: together we can do what no one can do alone. That's right! And when working as a team, even the smallest and weakest can turn out to be the decisive link.

Now ask them to count the number of characters in "The Turnip." People usually say there are six. But if you count the turnip as a character too, then there are seven. And the appearance of each new character makes you start the story from the very beginning. And by the time we get to the mouse, the story has already been learned! And there's no need to study or memorize anything on purpose.

So, to remember a topic, you need to repeat it seven times. Perhaps this is where the saying "Measure seven times, cut once" comes from?

In the same way, every new topic in a school lesson should be connected to the previous topics. And if things go right, there's no need to memorize: everything just happens naturally.

Grandma for Grandpa, Grandpa for the turnip!

7. What is counting?

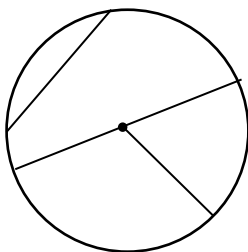
You have taught the child **to observe**, **describe** what is observed, formulate **a definition**, and assign **a term** to it. What's next?

Next, we'll learn to count.

Of course, you know how to count. But in order to teach a child to count, you must first understand and recall what it means, and then learn how to explain it...

But first, there needs to be a need for counting. Otherwise, it will be hard to explain why we should spend time and effort on this process. Try to clearly explain to yourself how the need for counting arose, and compare it with what comes next.

So, where did we leave off? We ended up with the following diagram:



The child already knows what a circle, a chord, a radius, and a diameter are. Now ask: are there any identical terms in the diagram? At first, it will seem that there aren't. But then the child will distinguish between chords, line segments, radii, and points. A set of identical terms forms *a group*.

A new question immediately arises: how many are there in the group? How do you convey information about quantity to another person, i.e., how do you describe the quantity?

Place three pencils on the table. If another person is nearby, you simply show them the pencils, and they see for themselves how many there are. But if you've gone outside and someone asks how many pencils were on the table, you need to describe their quantity, and to do that, you need **to count** them.

In the word “count” (“sčet”), “s” is the prefix; “chet” is the root. In the old days, the word “cheta” meant a pair. That is, they counted in pairs, two at a time. So many pairs. At first glance, it seems there's nothing complicated here. Everyone knows how to count. But let's see if that's really the case.

Place three pencils in front of your child and ask them to count them. Most likely, they'll immediately say “three.” But ask them to count them anyway. Count these specific pencils. They will count, pointing to each pencil and saying, “one, two, three.” As soon as they point to the third pencil and say “three,” pick up that pencil and ask, “How many is this?” They will say, “one”¹. But why did they just say it was “three”?

Have him count again. He will start pointing to the pencils one by one and saying, “one, two, three.” As soon as he points to the second pencil and says “two,” stop him, pick up that pencil, and ask, “How many is that?” He'll say

¹ Often, pointing to the third pencil held up, the child says, “Three.” That's how he was taught.

it's "one." But why did he say it was "two" before that? Have him count again.

The child is already somewhat confused. It seems like they're doing everything right, but the counting just isn't working out. Then, in desperation, they'll say, pointing to the pencils one by one: "one, one, one."

Now it's your turn to exclaim in frustration: "But where's the count of ? What if there were a hundred pencils?"

But remember how your little one learned to walk. You didn't yell at him when he couldn't do it. You helped him again and again, comforting him after every failure. And how happy you were when they took their first step on their own! And you didn't hide your joy from them. And the little one felt it and was eager to take another step... Even if your child is now ten, fifteen... They're still that same little one. They still need your comfort and your joy...

So what is the " " counting method?

Show your child a pencil and say, "This is one pencil." Then ask, which word did you say that was unnecessary? It's good if the child quickly figures out that the word is "one." "Pencil" is already singular. There's no need to add that it's one. Now pick up three pencils and say that these are pencils. Then pick up five pencils and say that these are pencils too. But the number of pencils in the first and second cases is different. If we say "pencils" in the plural, we need to describe their quantity. Describing the quantity of " " is *counting* . But describing the quantity of what and where?

Take three pencils, two pens, and one marker. Ask, “How many pencils are there?” The child will answer, “Three.” “And how many pens?” “Two.” “And how many markers?” “One.” “And how many items in total?” “Six!”

Why were the answers different each time?

Because we were talking about different categories: first pencils, then pens, then markers, and finally, objects.

*A set of objects with the same name is called a **group**.*

*The combination of objects with the same name is called a **grouping** .*

*A description of the number of objects in a group is a **count**.*

*The term denoting the result of counting is a **numeral**.*

*The symbol denoting a numeral is a **digit** .*

It should be added that there are countable groups and uncountable groups. Uncountable groups, for example, include the stars in the sky or the leaves in the forest.

How do you describe the number of items in a countable group? The simplest way to describe the number of items in a group is to count on your fingers.

When your child was three years old, he knew how to count on his fingers. Check to see if he’s forgotten how to do it. Place three pencils in front of him and have him count them on his fingers. He’ll confidently start bending his fingers and saying, “One, two...” Stop them and point out that they’re counting their fingers, but they should be counting the pencils . They’ll start pointing at the pencils with their fingers and saying “one, two...”. Stop him again and point out that he’s counting with his fingers, but he

should be counting on his fingers. The child is confused. Remind him that those who don't yet know numbers count on their fingers. That is, counting on fingers is done silently: set aside one pencil—fold down one finger; set aside another pencil—fold down another finger... Until you have a group of fingers equal to the group being counted. We demonstrate this by saying, "That's how many!"

How many pencils can you count on the fingers of one hand? That's how many! We show all the fingers on one hand. That's five. Can you count more than five on the fingers of one hand? Yes, you can. How?

Is the child thinking? Let's help them. We suggest that they use their thumb to mark the joints of the four remaining fingers. The resulting number is called *a dozen*! Today, that's twelve. 12 o'clock on the clock, 12 months in a year, 12 brothers in fairy tales... All from a dozen.

And how many can you count on the fingers of both hands? The child will immediately say twenty-four. That is, two dozen. But in the old days, people knew how to count up to five dozen using the fingers of both hands. They counted a dozen on one hand and bent one finger on the other; they counted another dozen and bent another finger... Five dozen used to be called *a "kopa."* Today, this number is called sixty.

In the old days, the sexagesimal counting system was widespread. It has remained, for example, in clocks: 60 seconds—1 minute, and 60 minutes—1 hour. The sixtieth part of a whole was called a kopeck—from the word *kopa*. Perhaps that is why the hundredth part of a ruble is still called a kopeck.

But can you count more than a kop on your fingers? Yes, you can. Let the child figure out how. If they can't figure it out for a long time, give them a hint: when counting a dozen on one hand, mark the second hand not with a whole finger, but just with a phalanx. This gives us a dozen dozen. This quantity was called *a gross*. Today, that's one hundred forty-four. There was also a term for a dozen dozen dozen: *a "mass."* Today, a mass is 1,728.

But in school, we haven't counted on our fingers for a long time. Remember how we counted in first grade? We counted with counting sticks. Notice, we didn't count the sticks, but with the sticks. How does that work?

"Kids, there are a lot of birds in the picture. Count them. Here's how to count: one bird—set aside one stick; another bird—set aside another stick... and so on, until we've counted all the birds. Now, show me how many birds there are in the picture? That's right, the same number as the sticks you have. You can take the sticks home and show them to your mom. Then your mom will know how many birds were in our picture."

To count means to describe the number of items in a given group.

But what if there are a lot of units? Just look at how the father sent his son to count the sheep in his flock, and the son brought back a whole bag of sticks... 3,457 of them. For this case, the first-grader's set includes multicolored sticks. We'll represent ten white sticks with one red stick, ten red sticks with one blue stick, and ten blue sticks with one black stick. So, the son needs to bring his father just three black sticks, four blue sticks, five red sticks, and

seven white sticks. All he has to do is remember the colors...

True, if the son accidentally drops one black stick along the way, he'll be off by a thousand sheep.

What if we make notches on a stick? Ten notches equal one cross. And ten crosses.... *Numbers* are already appearing (Roman numerals, for example)—symbols representing the quantity of objects of the same kind in a group.

Today, instead of using sticks or notches, we use *numerals* to describe the result of a calculation.

Numerals are terms that denote the number of units in groups.

Without the term “ ” and the “group ,” the numeral has no meaning.

Numerals can be represented by symbols—*digits*.

A group and a numeral can be visualized. The number and a set (as a mathematical term) cannot be visualized.

Try to visualize the number 5. You won't be able to. The symbol “5” is not a number; it is a numeral . It can also be written in Roman numerals as V. If you visualize five objects, then these are simply five objects (i.e., the set), not a number.

But you can perform operations on numbers, count using numbers, and then write down the result using digits or numerals.

At first, a child learns to group objects of the same kind. Then they learn to distinguish them by quantity. Then they memorize the names of quantities—numeral words. For example, today we are learning to recognize groups of

three objects of the same kind. There are three pencils on the table—that's *a group* of . Find some more groups of three in the room. That's right, three flowers in the vase, three heroes in the picture, three chairs in the room...

Numbers that describe the number of units in a group are called **natural** numbers. In fact, in nature we cannot observe three and a half sparrows on a branch or minus six horses in a meadow. Natural numbers are all positive integers from one to infinity. An example of natural numbers is Roman numerals. Here, all numbers are positive, integers, and there is no zero.

It takes time and considerable effort to achieve the ability to automatically distinguish and recognize groups based on the number of items in them and to memorize the corresponding numeral. Try to quickly learn to count in German, Chinese, or Japanese. And not just recite the numerals in order, but quickly answer the question of how many objects are being shown to you.

So, the main thing is not to rush, because your little one already knows how to count and, when asked how old they are, proudly holds up three fingers!

8. What are addition and multiplication?

Groups can be combined. If groups consist of identical terms, combining them results in a single new group. Combining groups with identical terms is called *addition*. If the terms are not identical, no new whole group is formed. Addition does not occur.

If you combine a group of five pencils with a group of two pencils, you'll get a new group of seven pencils. This is addition. And it doesn't matter whether we're talking about groups of pencils, pens, or chairs... As long as they're groups of items *with the same name*.

Addition makes counting much faster!

The symbol for adding groups is the “+” (plus) sign.

In fact, let's say we have a group of two pencils and a group of three pencils. Since the terms in both the first and second groups are the same, we can combine these two groups into one. We'll get a new group. It has five pencils.

Suppose we have a group consisting of three pencils and a group consisting of two pens. Since the terms in the groups are different, combining them does not form a new whole group. This is not addition. To combine them into a group, we need to find a term *common* to both groups: “.” If we say that the first and second groups consist of *objects*, then addition is possible: we get a group of five objects.

So, only terms of the same name can be combined into a group. Consequently, ***addition*** is the combination of groups of the same name.

We have reached a very important point. First, we need to ensure the child understands and can reproduce the expression: *adding **a group** of two pencils to **a group** of three pencils results in **a group** of five pencils.*

Once the child can say this fluently, you can move on to the abbreviated version: *adding the group “two” to the group “three” makes a group of “five.”*

And finally, once this formulation has been mastered, you can move on to the final version: *three plus two equals five.*

We will call the result of the addition ***the sum***.

Groups that are combined are called ***addends***.

Let's say we have a group of five pencils. If we remove two pencils from it, we'll be left with a group of three pencils. That's subtraction.

Subtracting the group “two” from the group “five” gives us the group “three.”

Final statement: *five minus two equals three!*

So, removing a group from a group is called ***subtraction***.

The result of subtraction is called ***the difference***.

And what is multiplication?

Let's consider the sequential addition of groups with an equal number of units in each.

For example, let's add a group of “three” to the group “three,” then another group of “three,” then another group of “three.” We have a sum of equal terms. There are four equal groups.

We will call the addition of equal addends ***multiplication***.

The numeral denoting the number of equal addends is called *a multiplier* . That is, *the multiplier* denotes the number of equal addends.

The sum of equal addends, or the result of multiplication, is called *the product*.

Multiplication is denoted by either a cross “ $\times$ ”, a dot “ $\bullet$ ”, or an asterisk “ $*$ ”.

The multiplication sign is used when the multiplicand comes first and the multiplier second: 3×4 means three times four, i.e., $3 \times 4 = 3 + 3 + 3 + 3 = 12$.

The dot is used when the multiplier comes first and the multiplicand second: $3 \bullet 4$ means three times four, i.e., $3 \bullet 4 = 4 + 4 + 4 = 12$.

The asterisk is usually used on a calculator when it doesn't matter whether the first number is the multiplicand or the multiplier: $3 * 4$ can mean either 3×4 or $3 \bullet 4$.

It is convenient to demonstrate the multiplication process, just like addition and subtraction, using the same sticks. Ask the child to demonstrate multiplying different numbers using the sticks. They will find this easy.

But will they be able to demonstrate division?

9. What is division?

What is 8 divided by 2?

The child will immediately answer, “four.”

Lay out eight sticks in front of them and ask them to divide them into two. They will most likely split the whole group into two parts and say, “Here—four.”

And you ask, “Four of what?”

They’ll point to one part, which has four sticks, and say, “Here, four sticks.”

– And how many are in the other part?

– Four as well.

– So, how many in total?

– Eight.

– So, we see that eight divided by two is eight?

– No, this one group is four.

– But that’s the result of subtraction. If you subtract four sticks from eight sticks, you’re left with four sticks. But we’re left with eight sticks, or two groups of four... There aren’t four on their own. So where’s the result of dividing by two?

– ?

What should we do?

If you divide anything into two parts, you get two parts. Where did the four come from?

It turned out to be a very interesting conversation.

So what is division?

So, dividing into parts is different from the arithmetic operation “division.” Dividing eight into two means breaking down the group of “eight” into equal parts. Into parts of “two.” You’ll get four equal parts of “two.” We

break down a group of 8 sticks into groups of 2 and get 4 equal groups.

The child will be amazed. They'll watch you as if you were a magician or a wizard who suddenly came to their aid in their battle against evil sorceresses!

What am I talking about?

From the very first grade, the child knew: anything divided by two equals two. And divided by three equals three. So, eight divided by two equals two, and divided by three equals three. Three parts! But the evil "teacher" screamed that no! That he was wrong!! That he was an ignoramus!!! Somehow, in some unimaginable way, she managed to make it so that eight divided by two equals four. And by three, she kept shouting, eight doesn't divide at all!

And here you are saying such clear, simple things. The witch's spell is beginning to break.

A group can only be broken down into factors. Breaking a group down into equal factors is called *division*. This operation is the inverse of multiplication. But why not say "divide **by two**" instead of "divide **into two**"? It would be clearer. They say that's just how the tradition developed. Well, let's follow tradition.

Multiplication is the sum of equal terms

Division is the factorization into equal factors.

The values of the equal terms into which the dividend is divided are called *divisors*.

The number of resulting equal terms (equal parts) is *the quotient*.

Consequently, the quotient denotes the number of divisors into which the dividend is broken down, or the number of resulting equal terms, or the number of result-

ing equal parts... Quotient. The resulting equal parts are called shares.

For example, dividing twelve by three means calculating into how many shares of “three” each the group of “twelve” can be divided. We get four shares of “three.” Here, the quotient is four.

What is called *the quotient* in division is called *the factor* in multiplication.

From this, it is clear why you cannot divide by zero: you cannot break down a given group into factors that do not exist!

Now ask yourself: how many sticks will there be in each part if 12 sticks are divided into three equal parts? The answer will again be “four.” But now these are no longer four equal parts. There are four in each part. And this is not a quotient. So what is it?

We have three ways to divide the group into parts.

In the first option, we need to calculate how many parts there will be. For example, dividing 12 into three parts will result in three parts. This division is called ***division into parts***. Everything is clear here—dividing anything into three parts will result in three parts.

In the second case, you need to calculate how many equal terms you will get if you know how many units there should be in each. For example, if you divide 12 into terms of three, you will get four equal terms. This type of division is called ***division by content***. The divisor precisely denotes the content of each term. The result of division by content is called *the quotient*. That is, the result shows how many equal terms (equal parts) will be obtained, each with a known equal content.

Division by content is also called *subtraction* by comparison. In fact, to divide 12 by 3, you can successive-

ly subtract 3 from 12 and count the number of threes obtained.

Division by content is denoted by a colon — “:”. For example, $12:3=4$.

In the third case—the reverse—we need to calculate the content of each component if we know into how many equal parts the dividend is divided. Equal parts are called *shares* here. The divisor in this case denotes the number of shares. For example, if 12 is divided into three shares, each will contain four units. This results in a *multiple* comparison—that is, 12 is three times 4. This division is called ***division by parts***. The result of division by parts is called *the quotient*. **The quotient shows how many units of the dividend there are per unit of the divisor.** Division by parts is denoted by the fraction bar “—” or the slash “/”.

For example, $\frac{12}{3} = 4$ or $12/3 = 4$. This means that if you divide twelve into three parts, each part will contain four. Or, twelve is four times three. Or, for every unit of the divisor “3,” there are four units of the dividend “12.”

As we can see, the results in the second and third methods are the same. Therefore, when performing abstract division of numbers, a combination of both symbols is often used as the division sign, i.e., “÷.” This symbol is called the “obelus.”

Consequently, $12\div3=4$ means both $12:3=4$ and $12/3=4$. In the first case, the result is the quotient, and in the second, the ratio.

When solving problems at the initial stage, it is advisable to distinguish between obtaining the quotient and the ratio. For example.

Problem 1. *Eight nuts were divided into baskets, with two nuts in each. How many baskets were there?*

Solution: $8 : 2 = 4$ (baskets)

Answer: There were four baskets.

Problem 2. *Eight nuts were divided equally into two baskets. How many nuts are in one basket?*

Solution: $8 / 2 = 4$ (nuts)

Answer: There are four nuts in one basket.

If we use the ":" symbol, the answer is *four baskets*. But if we use the "/" symbol, the answer is *four nuts*. In the first case, we get the quotient, and in the second, the ratio.

Here's another example. Let's divide 51 by 4.

We write the division as follows:

$52 : 4 = 13$ — 13 groups of 4 each;
 4 this is the quotient.

12

 12

0

We write the division by fractions as:

$52 \overline{) 4}$
 4 $\overline{) 13}$

12

 12

0

— i.e., by one divisor

there are 13 units of the dividend;

—that's the ratio.

Very often, children assume that the second type of division is simply column division. But, as we can see, both are column divisions here. The mathematical result of the division is the same, but the physical meaning is different.

10. What are fractions?

Place seven sticks in front of the child and ask how many equal parts they see. After a brief pause, they will say, of course, seven.

Therefore, dividing our group by one means getting seven parts of one each. The whole group “7” consists of seven equal parts. One part will be one of seven equal parts.

Equal parts of a whole group are called *fractions*. Therefore, one stick in our group is one-seventh of that group.

Now, ask him to divide the seven sticks into three. He will say that seven cannot be divided by three. But let him try anyway.

He will start breaking the group of seven sticks into groups of three sticks. This will result in two groups of three, and one stick will be left over. A remainder has been produced... This is a fraction of a whole. How can we describe the size of this remainder now? Point to one stick and ask, what fraction of the group of seven is this? Confusion again. But, most likely, remembering the previous conversation, the child will say that this is one-seventh. And they will be right.

But we no longer have a whole group of seven sticks. We have the result of the division—the quotient. That’s two separate whole groups of three sticks each. And there weren’t enough sticks for a third group of two. There’s only one out of three. So, the remaining single stick is one-third.

Note that every time you get a remainder, you need to determine what fraction it represents of the whole num-

ber of the addend, not the divisor. And the number by which the entire group is divided is called *the divisor*. So the remainder is a fraction *of the divisor*.

Let's demonstrate this same operation using circles in a notebook. Let's divide 7 circles into groups of 3. The first and second groups each have three, while the third has only one—two are missing. There is only one out of three. So let's write it down:

$$\begin{array}{c}
 \overbrace{\bullet \bullet \bullet}^3 \quad \overbrace{\bullet \bullet}^3 \quad \overbrace{\bullet \times \times}^1 \\
 \underbrace{\hspace{1.5cm}}_3 \quad \underbrace{\hspace{1.5cm}}_3 \quad \underbrace{\hspace{1.5cm}}_3
 \end{array}
 \qquad 7:3 = 2\frac{1}{3}$$

Ask to divide the group "7" into groups of four. The divisor is "four." The result is one whole term "four" and three sticks in the remainder. Since our divisor is "four," one stick is now one-fourth of a whole. Therefore, the remainder is three-fourths

The numeral denoting the number and size of the parts is called *a fraction*.

The numbers indicating the number of parts and their size are separated by a fraction bar. That is, a fraction is a ratio, the result of dividing into parts. The number above the fraction bar is called the "numerator," and the number below the bar is called the "denominator." The denominator indicates the size of a part, and the numerator indicates the number of these parts.

Now ask the child, how many halves are there in total?

They will think about it.

Ask, "How many halves are there in a whole apple?"

I've gotten all sorts of answers to this question: from 1 to 8 halves...

To us adults, it all seems so obvious. Children are just getting used to the new terms.

So, now that we've established there are indeed two halves, show how one half is denoted. It is denoted as: $\frac{1}{2}$.

It is read as: one-second (part). This is the notation for one half. Half of what?

We know that there is no such thing as just a half. There is only a half of something. In our case, it is half an apple. In general, it is half of a group.

Let's continue. How many thirds of an apple are there in an apple?

That's right, three.

And how do we denote one of these thirds?

Let's denote it as: $\frac{1}{3}$ (one-third).

One-third of what? A whole apple.

And how many fourths of an apple are there in an apple? Four. And how do we denote one of these fourths?

It will be denoted as $\frac{1}{4}$ (one fourth).

One-fourth of what? A whole apple.

And into how many fifths can an apple be divided? Of course, into five. So, one of these five parts will be denoted as $\frac{1}{5}$ (one-fifth).

One-fifth of what? An apple.

Can we continue counting like this— —until the child finally understands what parts of a whole are and how they are denoted? And here it is very important to

explain that there are no such things as isolated fractions. There is no such thing as simply “two-thirds”! There are only two-thirds of something.

If we write the fraction $\frac{3}{5}$, it means not just three-fifths, but three-fifths of something. In this case, three-fifths of one whole, i.e., of one. It is assumed that the original notation was: $\frac{3}{5} \cdot 1$. But we do not write the one.

Let’s go back from apples to sticks. Let’s divide the group of “eight” sticks into groups of “three” sticks. This will give us two complete groups of “three” and two sticks left over. How do we represent the remainder? Since we divided into groups of three, the resulting single whole group consists of three sticks. Therefore, one stick represents one-third of a whole. We have two sticks left over, so the remainder is two-thirds.

With group fractions, i.e., with fractions, you can perform the same arithmetic operations as with whole groups. You can add, subtract, multiply, and divide them. The main thing is that the fractions must be *equal*! Only in this case do they form new groups.

So, you can only add and subtract fractions that are equal in size. The number of parts is shown by the numerator, and the size of a part is shown by the denominator. And again, remember that fractions are equal parts.

11. Adding fractions

What will you get if you add one-fifth to one-fifth? Clearly, you'll get a group of consisting of two-fifths, i.e., two-fifths or $\frac{2}{5}$.

How many fifths of an apple will there be if we add two more fifths to one-fifth? Of course, three-fifths.

$$\frac{1}{5} + \frac{2}{5} = \frac{3}{5}.$$

Now we can solve the "examples":

$$\frac{2}{5} + \frac{2}{5} = \quad \frac{3}{7} + \frac{2}{7} = \quad \frac{3}{8} + \frac{1}{8} = \quad \frac{1}{6} + \frac{1}{6} =$$

In the same way, you can subtract the fraction from the fraction

$$\frac{3}{7} - \frac{2}{7} = \quad \frac{3}{8} - \frac{1}{8} = \quad \frac{5}{6} - \frac{1}{6} = \quad \frac{3}{4} - \frac{1}{4} =$$

It's all simple and straightforward.

Now surprise your child with a question: how much is left if you subtract one-fourth from one?

The child will think about it.

Ask right away, "How many quarters are there in a whole?" Help them realize that a whole is a group of and can be divided into any number of parts. In this case, we're subtracting one quarter. Therefore, we must first divide the whole into four parts. That gives us four quarters.

And how much did we subtract? One fourth.

And how much is left? Three-fourths!

Let's write it down.

$$1 - \frac{1}{4} = \frac{4}{4} - \frac{1}{4} = \frac{3}{4}$$

And how much will there be if we subtract two-sevenths from one?

And how many sevenths are there in one? That's right, seven sevenths. And how much did we subtract? Two sevenths. And how much is left? Five sevenths.

$$1 - \frac{2}{7} = \frac{7}{7} - \frac{2}{7} = \frac{5}{7}$$

And what will you get if you subtract five-ninths from one?

And how many ninths are there in one? Nine. And how much was subtracted? Five. How much is left? Four! Four-ninths.

$$1 - \frac{5}{9} = \frac{9}{9} - \frac{5}{9} = \frac{4}{9}$$

Point out to the child that every time we subtract the fraction from one, we are essentially subtracting the numerator from the denominator!

Try solving some more examples:

$1 - \frac{3}{7} =$	$1 - \frac{8}{9} =$	$1 - \frac{12}{13} =$
$1 - \frac{4}{5} =$	$1 - \frac{1}{8} =$	$1 - \frac{20}{27} =$
$1 - \frac{2}{7} =$	$1 - \frac{3}{5} =$	$1 - \frac{122}{127} =$

It's easy and simple again!

Make the problem more challenging. Ask, "What happens if you subtract three-sevenths from two?"

$$2 - \frac{3}{7} =$$

The child is thinking again.

It's good that they're thinking. They need to think as often as possible, recognizing new types of problems. And they need to find the solution on their own as often as possible. This will teach them not to be afraid of new problems.

Let's break down two whole units into one unit and another unit. We'll subtract three-sevenths from one unit, which gives us four-sevenths, and then add the second unit, which gives us one whole plus four-sevenths.

$$2 - \frac{3}{7} = 1 + 1 - \frac{3}{7} = 1 + \frac{4}{7} = 1\frac{4}{7}.$$

Come on, let's do some more.

$$5 - \frac{2}{9} = 4 + 1 - \frac{2}{9} = 4 + \frac{7}{9} = 4\frac{7}{9}.$$

Come up with about ten more examples...

Better yet, have them come up with a few examples and check whether you solved them correctly.

Very good!

Now let's check how well the child understands what they're doing.

Let's review the fractions from the very beginning. Ask them to demonstrate adding and subtracting fractions using sticks again. Most likely, they'll exclaim that this can't be done with sticks, since the sticks are whole. They might even try to break a stick into pieces...

Stop them and remind them that we're talking about groups. Fractions represent the size and number of parts of a whole group. A whole group can consist of any num-

ber of sticks, and each stick is a part of the whole group. Therefore, there's no need to break a stick into pieces. You can assemble a group in advance that's already divided into the required number of parts (sticks).

For example, lay out four sticks on the table. Put them together and say that this is one whole group, i.e., this is a unit. (Рис. 5).

If four sticks are one whole group (a unit), then what does a single stick represent? Of course, one-fourth of the whole group (the unit). That is, in this case, a single stick no longer acts as a unit, but as one-fourth of a unit.

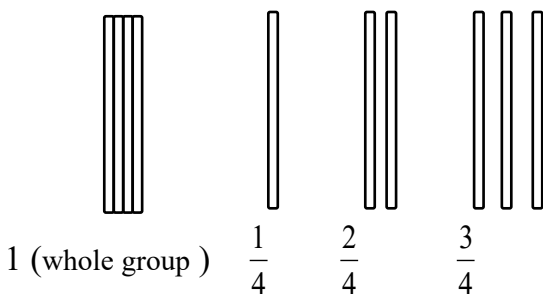


Рис. 5

How can we now demonstrate the addition of fractions?

It's very simple. Add one more fourth (1 bar) to one fourth, and we get two fourths. Add one more, and we get three fourths.

And how do we show one-fifth?

Let's take a group of five sticks and say that this is one whole group. Then, one stick will be one-fifth of this group.

Of course, we could sacrifice a stick, break it into pieces, and demonstrate fractions ... But, it's a shame to break it. And besides, in this case, the child gets used to

the idea that a fraction is a part of only a single unit. Later on, it will take a lot of effort to teach them to take a fraction not only from a single unit, but also from a group, i.e., an arbitrary number.

In fact, a fraction represents the value and the number of parts of a given group.

Suggest dividing the group of eight sticks into pairs again (dividing eight by two). This will result in four pairs.

Each resulting group is one-fourth of the original group "eight".

And how do we indicate what fraction this represents of the group "eight"? Of course, with a fraction: $\frac{1}{4}$.

One-fourth of what? Of course, the group "eight."

How many fourths are there in the group "eight"? Four fourths.

So, dividing eight by two gives four parts of two each.

Now ask to divide eight by three...

We break down the group of "eight" into equal parts of "three." We end up with two whole parts of "three" and one partial part of , which has only two sticks.

Ask, "How many sticks should there have been in the last term?" He'll say, "Three." But how many do we have? Only two. And what should we call this non-integer group? Let's call it "two-thirds."

So, we have two whole terms of three and two-thirds of this term, or "two and two-thirds ." This is a *mixed number* . It contains both a whole part and a fractional part. They're mixed together...

On the other hand, each whole group consists of three sticks. What do we call one stick? That's right—one third. How many such sticks are there in “two whole two-thirds”? Correct, eight. Eight of what? Thirds.

That is, $2\frac{2}{3} = \frac{8}{3}$. Here, the numerator is greater than the denominator.

Fractions of the form $\frac{a}{b}$, where the numerators are greater than the denominators, are called *improper fractions*.

Now practice dividing numbers using sticks, forming mixed numbers, and converting mixed numbers to improper fractions () and vice versa. You already know how to do this. You'll be able to do it.

For example, how do you calculate the difference

$$1\frac{2}{7} - \frac{6}{7} =$$

A child might say that you can't subtract five sevenths from two sevenths. That's right. But you can subtract it from one. That leaves one seventh. And we still have two sevenths left. So, the total will be three sevenths.

Or let's divide one whole into seven sevenths and add it to two sevenths. That makes nine sevenths. Subtracting six sevenths from nine sevenths gives three sevenths.

$$1\frac{2}{7} - \frac{6}{7} = \frac{9}{7} - \frac{6}{7} = \frac{3}{7}$$

It will be interesting to see how a child, while calculating the fractions $\frac{a}{b}$, checks the answer using the counting method on sticks.

12. How do you add fractions with different denominators?

Once the child is completely confident that they can do everything now, give them the following:

$$\frac{1}{2} + \frac{1}{3} =$$

Some children will pause to think, while others will confidently write $\frac{2}{5}$.

First of all, it's important to remind them that only identical, equal fractions can be added or subtracted. In the last example, the fractions are not equal.

How do you add one-third to one-half?

Draw half an apple and a third of an apple on a sheet of paper. Show that they are different in size and that the result of adding them together will be something that cannot be described immediately. Or, again, you end up with one-half and one-third.

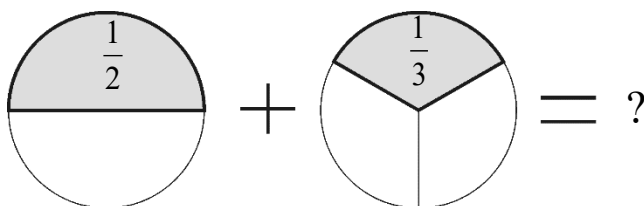


Рис. 6

Parts of different sizes do not add up to a single group. Just as objects with different names (terms) do not add up to a single group.

What should we do?

Remind them that to combine two groups with different terms into a single group, we selected a common term for them: .

We'll do the same with different fractions. We'll find a common fraction for them. That is, a fraction that would fit into both of our unequal fractions exactly times.

First, ask which fraction is smaller: one-half or one-third?

Children often say that one-half is smaller. Once you've confirmed that one-third is actually smaller, suggest it as the common fraction.

On the right, the whole is divided into three parts. On the left, it is divided into two. If we also divide the left side into three parts, we can see that these new three thirds will not be evenly distributed among the original two halves. Therefore, one-third cannot be a common part here.

Let's divide one third on the right into two. This will give us two sixths. Why? We have three thirds in total. If we divide each one in half, we get six parts of the whole. Two for each third.

Could the sixth part be the common part?

Let's divide the whole on the left into six. We'll also get six sixths. The new sixths will be distributed among the old second parts , equally—three sixths for each. That's exactly what we need!

Now we have common parts for one-second and one-third. These are the sixths.

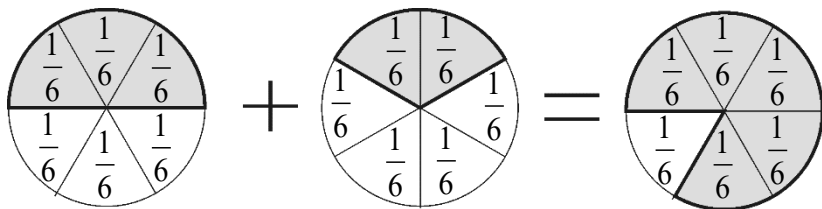


Рис. 7

One second contains three sixths, and one third contains two sixths. In total—five sixths!

Therefore,

$$\frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

Here is the result—a bit strange, perhaps.

Nevertheless, it is the correct result.

But we're not going to draw circles every time we add different fractions.

No, of course not. A simpler method has been found for adding (and subtracting) fractions with different denominators.

Let's look at our sum again:

$$\frac{1}{2} + \frac{1}{3} =$$

One-third is less than one-half. Three-thirds cannot be divided between two halves; that is, three does not go into two. Therefore, one-third cannot be a common denominator. Let's divide one-third by two, that is, multiply the larger denominator by two. We get one-sixths. Six sixths can be divided between two halves. That is, each

half receives three sixths. Consequently, one sixth is the common fraction. Six is divisible by both two and three.

$$\frac{3}{6} + \frac{2}{6} = \frac{3+2}{6} = \frac{5}{6}.$$

Let's look at another example.

$$\frac{1}{8} + \frac{1}{6} =$$

Here, the smaller fraction is one-eighth. One whole consists of eight eighths. They cannot be divided equally among six sixths. That is, eight is not divisible by six.

We multiply eight by two (i.e., we halve the eighth part). We get sixteenths. Sixteenths also cannot be divided equally among six sixths. Sixteen is not divisible by six.

We multiply eight by three (i.e., we reduce the eighth part by a factor of three). We get twenty-fourths. Twenty-four twenty-fourths can be distributed among six sixths. Twenty-four is divisible by six. This gives four twenty-fourths to each sixth part. And each eighth part, of course, gets three twenty-fourths.

Therefore, the largest fraction that fits an integer number of times into one-eighth and one-sixth is one twenty-fourth.

This common fraction is called the least common denominator.

We get

$$\frac{3}{24} + \frac{4}{24} = \frac{7}{24}$$

Let's do some more.

$$\frac{3}{5} + \frac{2}{3} =$$

Five doesn't go into three. We multiply five by two and get ten. Ten doesn't go into three. We multiply five by three and get fifteen. Fifteen goes into three. This is the common denominator, so the common fraction will be one-fifteenth.

$$\frac{3}{5} + \frac{2}{3} = \frac{\quad}{15}$$

To make the denominators of the first and second fractions equal, we need to multiply the numerator and denominator of the first fraction () by three, and those of the second by five. For convenience, let's write them as additional factors:

$$3) \frac{3}{5} + \frac{52}{3} = \frac{9+10}{15} = \frac{19}{15}$$

or

$$4) \frac{5}{12} + \frac{37}{16} = \frac{20+21}{48} = \frac{41}{48}$$

Calculate.

$$\frac{1}{5} + \frac{3}{4} =$$

$$\frac{1}{7} + \frac{3}{14} =$$

$$\frac{2}{9} + \frac{1}{6} =$$

$$\frac{2}{3} + \frac{1}{6} =$$

$$\frac{7}{8} - \frac{5}{6} =$$

$$\frac{7}{9} - \frac{1}{2} =$$

$$\frac{5}{7} + \frac{1}{21} =$$

$$\frac{3}{4} - \frac{2}{7} =$$

$$\frac{5}{9} + \frac{1}{4} =$$

So, we have learned how to add and subtract common fractions .

13. How do we multiply fractions?

Let's recall what it means to multiply a number by a number?

For example, multiplying 3 by 6 means taking six equal groups of three. That makes 18. Got it.

And what does it mean to multiply $\frac{3}{4}$ by 6?

Most likely, it means the same thing. You need to take six equal terms of three-fourths each.

$$\frac{3}{4} \cdot 6 = \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} = \frac{3+3+3+3+3+3}{4} = \frac{3 \cdot 6}{4} = \frac{18}{4}$$

Therefore, to multiply a fraction by a number, you must multiply the numerator of the fraction by that number. The result will be eighteen-fourths.

And what is 6 times $\cdot \frac{3}{4}$?

Changing the order of the factors does not change the product, i.e.,

$$6 \cdot \frac{3}{4} = \frac{3}{4} \cdot 6 = \frac{18}{4}$$

But we want to understand how this works within the framework of our definitions. What does it mean to multiply 6 by $\frac{3}{4}$?

Here, the multiplier is three-fourths. This means we need to take the addend six... how many times? Three-

fourths of a time. But how do we do that? How do we take three-fourths of a time? Taking one, two, or three times — it's clear how: we need to multiply. But three-fourths of a time? We also need to multiply. But how? How can we demonstrate this using sticks?

Here on the table we have six sticks. Multiplying by three means taking three equal groups of six. That makes eighteen sticks. That's clear.

Now, how do you multiply six sticks by three-quarters? Let's look at how to multiply a whole number by a fraction.

Multiplying by three-quarters means taking the six six times. But how do we understand that?

First, let's try to understand what it means to take one-fourth of a unit. Most likely, it means taking one-fourth of a unit.

We have a new term here—"to take." What does it mean?

We know that subtracting one-fourth from one whole gives three-fourths. That is,

$$1 - \frac{1}{4} = \frac{3}{4}$$

And what will we get if *we take* one-fourth from one whole?

It seems like there's no difference between "take" and "subtract." But that's just an illusion.

Subtraction results in a difference. That is, we must name the quantity that remains after subtraction.

But taking one-fourth of a whole means we have to name the amount we took...

How much and what did we take?

We've already noted that the fraction $\frac{1}{4}$ means not just one-fourth, but one-fourth of something. In this case, one-fourth of a whole. If you're told to take one-fourth of a whole, that means you must take one-fourth!

Taking $\frac{1}{4}$ of a whole will give $\frac{1}{4}$.

Taking one unit $\frac{1}{4}$ of a time also gives $\frac{1}{4}$.
one-4 th

Therefore, taking $\frac{1}{4}$ from 6 units will be 6 times more, since you need to take one-fourth from each unit. And this is nothing other than multiplication:

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{1}{4} \cdot 6 = \frac{6}{4}.$$

And taking one-fourth of six times will also be six times four:

$$6 \cdot \frac{1}{4} = \frac{6}{4}.$$

Let's try doing the same thing using sticks. We already know how to represent one-fourth. Take a group of four sticks—this is one whole group consisting of four one-fourths. Set aside three one-fourths. Now we need to take six of these parts... That makes six one-fourths.

Accordingly, if we take three-fourths of six, the result will be three times as much—eighteen-fourths.

So, we've figured out the difference between "subtracting" and "taking." To take a fraction of a number, you need to multiply that fraction by the number.

Consequently, multiplying a whole number by a fraction means taking a fraction of the number.

And how do you multiply a fraction by a fraction?

For example, how do you multiply three-fourths by five-sevenths? This means you need to take three-fourths of five-sevenths.

To calculate one-fourth of five-sevenths, you need to divide five-sevenths by four. And how do you divide fractions?

14. How do you divide fractions?

How do you divide a number by a fraction? For example, what does it mean to divide six by one-third?

As we already know, division is the breakdown of a group into equal parts.

In our case, we will break down the group “six” into parts of “one-third.” One whole is three-thirds. Six wholes is six times as much, that is, 18.

$$6 : \frac{1}{3} = 18.$$

To make the calculation easier, let’s remember that division is the inverse operation of multiplication. Therefore, we can replace the division sign with a multiplication sign and the divisor with the reciprocal of the divisor. The result will be the same.

$$6 : \frac{1}{3} = 6 \cdot \frac{3}{1} = 18.$$

Accordingly, dividing six by terms of three-fourths each will be... Four-fourths in one. This means that six whole units contain twenty-four-fourths. Dividing them into groups of three terms... yields eight groups, i.e.

$$6 : \frac{3}{4} = \frac{6 \cdot 4}{3} = 8.$$

But how do you divide one-third into two? Not into two parts, but into two. Division is the process of breaking a group down into equal parts. We need to break down the group “one-third” into parts of “two.” Parts of “two” of what? Two whole units. But we don’t even have a single whole unit!

To get at least one whole number term, we need three-thirds of a share. To get the desired “two” term, we need six-thirds of a share. But we really only have one-third of a share? One-third of the six represented shares... Therefore, the answer is one-sixth.

But calculating every time how many represented parts there are and how many of them are real is quite difficult.

Since one-third is one divided by three, dividing it again by two means multiplying the denominator by two.

$$\frac{1}{3} : 2 = \frac{1}{3 \cdot 2} = \frac{1}{6}$$

Or you can use the method of replacing division with its inverse operation (multiplication) and replacing the divisor with its reciprocal.

$$\frac{1}{3} : 2 = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$$

We divide a fraction by a fraction in exactly the same way. We multiply the fraction by the denominator and divide by the numerator. This results in the famous “inversion.”

$$\frac{5}{7} : \frac{3}{4} = \frac{5 \cdot 4}{7 \cdot 3} = \frac{20}{21}$$

15. What are decimal fractions?

Fractions such as with denominators of 10, 100, 1000, etc., are more convenient to write in a new form, separating the whole number from the fractional part with a simple comma. In this case, the denominator is not written in the fractional part; only the numerator is written.

But how do we know what the denominator was?

We decided that the number of decimal places should correspond to the number of zeros in the denominator.

For example,

$$\frac{25}{100} = 0,25; \quad \frac{7}{10} = 0,7; \quad \frac{251}{1000} = 0,251; \quad 3\frac{9}{10} = 3,9;$$

The fraction written in this new form is called a decimal. Performing arithmetic operations with decimals is much more convenient than with ordinary fractions.

For example, addition and subtraction. Here, there is no need to find a common denominator. The main thing is that when adding in columnar form, the whole part of the decimal fraction is aligned with the whole part, and the fractional part is aligned with the fractional part—that is, the decimal point must always be aligned with the decimal point:

$$\begin{array}{r} 23,456 \\ - 2,231 \\ \hline 21,225 \end{array} \quad \begin{array}{r} 23,456 \\ + 2,2 \\ \hline 25,656 \end{array} \quad \begin{array}{r} 13.45 \\ - 0.27 \\ \hline 13.18 \end{array} \quad \begin{array}{r} 0.406 \\ + 52.03 \\ \hline 52.436 \end{array}$$

And how do you multiply decimal fractions ?

First, simply multiply the numbers, ignoring the decimal points. In the product, count from right to left as many digits as there are in both factors combined, and

place a decimal point. Wherever there are not enough digits, insert a 0.

For example,

$$\begin{array}{r} \times 23.4 \\ \times \underline{3.1} \\ + 234 \\ \underline{702.} \end{array}$$

$$\begin{array}{r} \times 5.34 \\ \times \underline{0.11} \\ + 534 \\ \underline{534} \\ 0.5874 \end{array}$$

$$\begin{array}{r} \times 0.204 \\ \times \underline{0.3} \\ 0.0612 \end{array}$$

$$\begin{array}{r} \times 0.004 \\ \times \underline{7} \\ 0.028 \end{array}$$

And finally, division.

First, divide the decimal fractions, ignoring the commas. In the resulting quotient, separate from right to left as many digits as there are in the dividend after the decimal point, and place a comma; then move that comma from left to right by as many digits as there are in the divisor after the decimal point. Wherever there are not enough digits, we insert 0.

$$32.7 : 0.03 = 1090.0; \quad 0.327 : 0.3 = 1090.0$$

This procedure can be simplified. First, in the divisor, move the decimal point to the right until you get a whole number: . Then, move the decimal point in the dividend to the right by the same number of places. Next, divide as usual.

Children will learn to calculate quickly. But our task at this stage is to teach children the language of mathematics. Let the child try to reproduce the entire text provided. He knows what to say; he knows how to say it. But will he be able to speak freely?

16. And how do you convert a decimal fraction into a common fraction and vice versa?

It's very simple: we write it the way we hear it.

For example, how do you convert 0.7 into a common fraction? Here, it's seven-tenths.

So we write: $\frac{7}{10}$, i.e., $0.7 = \frac{7}{10}$

More. The fraction 4.13 is four and thirteen hundredths. So, we write it like this: $4\frac{13}{100}$, i.e., $4.13 = 4\frac{13}{100}$.

And how do we convert a common fraction to a decimal? It's just as simple. After all, we know how to divide and write the quotient as the decimal. So, we just need to divide the numerator by the denominator.

For example,

$$\frac{3}{4} = 3 : 4 = 0,75 \quad \text{or} \quad 5\frac{7}{8} = 5 + 7 : 8 = 5 + 0,875 = 5,875$$

Let's calculate the value of the expression $\frac{7}{8} - 2,5 \cdot \frac{1}{4}$

Option 1 — convert the fractions to decimals:

$$\begin{aligned} \frac{7}{8} - 2,5 \cdot \frac{1}{4} &= 7 : 8 - 2,5 \cdot (1 : 4) = 0,875 - 2,5 \cdot 0,25 = \\ &= 0,875 - 0,625 = 0,25 \end{aligned}$$

Option 2 — convert the fractions to common fractions

$$\begin{aligned}\frac{7}{8} - 2,5 \cdot \frac{1}{4} &= \frac{7}{8} - 2 \frac{5}{10} \cdot \frac{1}{4} = \frac{7}{8} - \frac{25}{10} \cdot \frac{1}{4} = \frac{7}{8} - \frac{25}{40} = \\ &= \frac{35-25}{40} = \frac{10}{40} = \frac{1}{4}.\end{aligned}$$

17. What are percentages?

From the set of decimal fractions, we selected those representing hundredths. They were called percentages. From the Latin *per cent* — “per hundred; one hundredth.”

The percentage symbol is abbreviated as “%”. This symbol literally means “divide by one hundred”. Therefore, in calculations, the percentage symbol “%” is replaced either by the divisor 100 or by the multiplier 0.01.

For example, $5\% = 5/100 = 0.05$ or $5\% = 5 \cdot 0.01 = 0.05$.

A percentage of a number is calculated the same way as any fraction of a number. You need to multiply the number of hundredths (percent) by the number.

For example, how do you calculate three percent of four (3% of 4)?

Let’s write this problem in mathematical terms, replacing the word “percent” with “hundredths.” We get:

$$0.03 \times 4 = 0.12$$

How do you solve simple percentage problems?

For example, what will the price of an item be after a 20% discount if the item originally cost 3,200 rubles?

First, we need to calculate the amount of the discount in rubles, i.e., we need to calculate how much twenty percent of three thousand two hundred is. Let’s write it down:

$$0.20 \times 3,200 = 640 \text{ (rubles)}$$

The new price will be this amount less, i.e.

$$3,200 - 640 = 2,560 \text{ (rubles)}$$

18. How do we translate the problem's text into mathematical terms?

If you place a group of three sticks and a group of five sticks in front of a child and ask which group has more sticks, they will, of course, easily show that the group of five sticks has more. But will they be able to write this down using numbers?

Some children already know the “greater than” or “less than” symbols and will confidently write: $5 > 3$ or $3 < 5$.

If the child writes this down, of course, you should praise them and immediately ask: by how much is the group of “5” greater than the group of “2”? Most likely, they will answer right away that it is by three.

And how do we write that down now?

Some children start writing it like this:

$5 > 3$ by 2...

Ask them to write down this expression without using prepositions, without using the Russian language, but using only the language of mathematics: only numbers and symbols. This will already cause some difficulty.

All numerals are related to one another. After all, they are all terms denoting the number of units in groups. The equal sign denotes this relationship. We'll use this: “five” is related to “two,” but is greater by “three.” This can be written very simply using numbers: $5 = 2 + 3$.

And how do we write that “six” is “three” less than “nine”? Here, the child will quickly write it down themselves: $6 = 9 - 3$.

Teach them to read numerical expressions in this way. This will be very useful when solving problems. After all, in most cases, solving a problem simply involves translating its text from language into the language of mathematics.

Now ask them to read the expression $7 = 2 + 5$ in three different ways.

The result should be:

- 1) “seven” equals “two” plus “five”;
- 2) “seven” is related to “two” and is “five” more;
- 3) “seven” is “five” more than “two.”

Saying these phrases aloud is very important. Ask the child to create their own numerical expressions and read them in three different ways.

Now we can move on to the next step. How do we write in numbers that “12” is three times greater than “4”?

Most likely, the child will already figure it out and confidently write: $12 = 4 \times 3$. That is, “twelve” is related to “four” and is three times greater.

And how do you write that “8” is less than “16” by a factor of “two”?

This won’t cause any difficulty at all.

Of course, $8 = 16 \div 2$. Or: “eight” is related to “sixteen,” but is “two” times smaller.

Take the time to practice with your child translating from the language of math into English and vice versa. After this kind of practice, math problems will be very easy to solve.

19. What is a formula?

We have learned to observe, describe what we observe, formulate definitions, assign terms to them, measure, distinguish between groups, and count.

What's next?

Let's not forget that every next step must be well-motivated. The child must understand the necessity of the next step; otherwise, they won't take it.

So, let's draw the familiar circle .

Let's draw a chord through the center of the circle. This is *the diameter* .

Let's connect a point on the circle to its center—this is *the radius* .

The lengths of the radius and the diameter can be measured. Let the child measure them with a ruler. This is easy.

Let's do the same with the other circle . Let's compare the two diameters. They differ in size. Nevertheless, they are diameters. The terms are the same, but the sizes are different.

Terms do not distinguish between sizes.

Let's denote the length of the diameter by the letter d , and the length of the radius. These are *parameters*.

For the first circle, the length of its diameter will be d_1 , and for the second, d_2 . And so on for any diameter of any circle: the general notation for the length of the diameter will remain the same (d); only the subscripts will change. Recall that d and r are parameters.

The parameter is the lettered notation of a quantity.

And *the value* is the result of a calculation or measurement.

Now let's note that the diameter consists of two radii. Therefore, we can write:

$$d = 2r.$$

This is our first formula: !

And don't worry that your child is too young to learn formulas. By third grade, kids are already comfortable working with variables. In other words, they already understand algebra.

So, what is a formula ?

A formula is a parametric representation of the relationship between quantities, or simply *a relationship between parameters*.

What is it used for?

The formula allows us to perform calculations.

What is a calculation?

A calculation is obtaining the value of a parameter not by measuring it directly, but by using its relationships with other parameters.

For example, if we have measured the radius, there is no need to measure the diameter. We can calculate it—that is, obtain it—by multiplying the radius by two.

Too simple. Why calculate the diameter when we can measure it too? Measuring is even more interesting...

That, or something like that, is what your child might say.

You would, of course, agree with that.

In fact, can you measure the length of a chord with a ruler?

Yes, you can.

Can you measure the diameter with a ruler?

Yes.

The length of the radius?

Yes, you can.

But can you measure the circumference with a ruler?

No!

So what should we do?

Let's denote the circumference by the *letter c*. The Latin letter *c*—the first letter in the English word “circle”—. So, a ruler is a straight line, and a circle is a curve. You can't measure it with a ruler. What else does a child need besides a ruler to measure the circumference?

If they guess that it's a string, that's great.

You can lay a string along the circumference, then pull it taut and measure it with a ruler. The length of the string will be equal to the circumference.

But, first of all, the measurements won't be entirely accurate. Second, a string isn't always helpful... What should we do?

We know that two radii fit within the length of the diameter. And how many diameters fit within the circumference?

We have the circumference measured using a string. Let's divide it by the measured diameter of the same circle. The result is approximately 3. You can do the same with circles of different sizes: the answer will be the same. In fact, the circumference of *any* circle contains approximately three times its diameter. To be more precise—3.14 diameters. And to be even more precise—3.141592653589793238462643... Rounded—3.14.

3.14... is the famous number . It is called **the number π** (pi).

Therefore, $s = \pi d$.

The number π is equal to the circumference of a circle with a diameter of one unit, or the number π is the circumference per unit of diameter.

This means that the rather tedious measurement of the circumference can be replaced by the relatively simple measurement of its diameter and multiplying the result by the number π (by 3.14). We *calculated* the value of the circumference by *measuring* its diameter.

But it is easier to measure the length of the radius than to measure the length of the diameter. To measure the length of the diameter, you need to align three points with a ruler—two points on the circle and its center—while to measure the length of the radius, it is enough to align only two. But we know that $d = 2r$.

Therefore, $c = 2\pi r$.

This is the formula for the circumference of a circle.

The formula allows us to calculate what is difficult to measure using what is easy to measure.

Ask the child to repeat the last expression. You'll find that they won't be able to do it. Make sure they memorize this expression and can say it without hesitation. Of course, you shouldn't force them. It will all work out on its own. After all, the child has understood exactly what needs to be said: the length of the diameter is easy to measure, but the circumference isn't. The formula allows... He himself will be somewhat confused by the fact that he can't say aloud what comes so easily in his head. As soon as he stumbles, just smile and ask him to repeat it again. You can also grab a stopwatch and have a contest:

who can repeat the entire sentence the fastest and without mistakes.

The formula does not eliminate the need to measure. But it allows you to replace a difficult or impossible measurement with a simpler measurement and calculation. A formula is a faithful friend and helper. Try to make it a friend and helper for your child, not a source of fear...

Give your child an old CD and a ruler. Let them figure out what the circumference of this disc is. Watch what and how they measure. Then have them write down the formula. Next, have them plug the measured value of the diameter into the formula. And let them calculate the circumference. They'll get $c = 37.68$ cm.

Now ask if they measured the circumference. They'll say yes!

Correct him. He didn't measure the circumference. He measured the diameter. And he *calculated* the circumference.

It is very important to show the difference between *measuring* and *calculating* from the very beginning.

20. What is a problem?

Yes, children solve “examples.” “Examples” is such a childish term... What kind of examples? What are they examples of?—they don’t know, but they solve them. But problems—those are too difficult. Children don’t like problems. They don’t like them because they don’t understand what’s required of them. And we’ll try to understand.

What is a problem? In English, *a problem* is a problem. This means we were doing some work; we managed to accomplish some things, but others we didn’t. A problem arose—a problem. And computational problems? We managed to measure or count some things, but others we didn’t. We need to compute it. How? That is the solution to the problem.

First, let’s look at a problem where, above all, we need to understand exactly what needs to be calculated.

“How many floors do you need to go up in the elevator to reach the 7th floor?”

Children often answer that you need to go up seven floors. But if you go up seven floors, which floor will you end up on? Of course, the 8th. But we need to be on the 7th.

Now let’s look at a problem with a simple calculation.

“The distance between floors in a nine-story building is 3 meters. How many meters do you need to go up in the elevator to reach the 6th floor?”

We already know that we need to go up 5 floors to reach the 6th floor. Therefore, we need to go up 15 meters.

Here's another problem.

"One boat can hold 8 people. What is the minimum number of boats needed to accommodate 43 people?"

Clearly, the people need to be divided into groups of 8. One boat per group. We divide 43 people into groups of 8 and get 5 groups of 8 people, with 3 people left over.

So, we need five boats for the five groups of eight people and one more boat for the three people. A total of 6 boats are needed.

Let's try a more complicated problem.

"One ticket booth sold 30 fewer tickets than the other. How many tickets were sold at each booth if a total of 480 tickets were sold?"

An experienced person would say that if there were two ticket booths, you need to divide the total number of tickets in half, then take 15 from one half and add it to the other, i.e.

$$480/2=240.$$

$$240 - 15 = 225 \text{ — tickets sold at the first booth.}$$

$$240 + 15 = 255 \text{ — tickets sold at the second ticket window.}$$

But the child is upset.

"Teacher," he'll say, "that's not the right answer. We need a solution that follows the rules."

Your objections—that the main thing is to solve the problem, and the method doesn't matter—won't comfort him.

What should we do? We just need to remember that these are so-called "problems with x's." That is, we denote something with the letter x and set up an "equation" that we already know how to solve...

Let's give it a try.

Let's say x tickets were sold at the first ticket booth. Then thirty more were sold at the second one, i.e., $(x+30)$ tickets. A total of 480 tickets were sold, which we'll write as:

$$x + (x + 30) = 480$$

We solve the "problem," i.e., the equation :

$$2x + 30 = 480.$$

Let's subtract 30 from both sides.

$$2 \times 30 = 480;$$

$$\begin{array}{r} -30 \quad -30 \\ \hline \end{array}$$

$$2 \times 0 = 450$$

Now let's divide both sides of the equation by 2.

$$\begin{array}{r} 2x \quad = 450 \\ \hline 2 \quad \quad 2 \end{array}$$

$$x = 225$$

225 is the number of tickets sold at the first ticket window.

$x + 30 = 225 + 30 = 255$ — this is the number of tickets sold at the second ticket window.

“But we don't know how to subtract and divide equations,” a child might say to you.

Yes. Indeed. In the traditional school system, they use the method of moving variables from one side of the equation to the other with the corresponding change in signs... What “moving” means and why the sign changes remains a mystery to most people.

But there are even more mysteries here. Ask them, “What is an equation?”

21. How do you solve equations?

What is an equation ? An equation — that's where the x is! And what is x ? It's the unknown. So where there's an unknown, there's an equation ? It's unclear.

We need to figure out where equations come from and what it means to solve them ?

We already know what *a quantity*, *a parameter* , and *a formula* are.

A quantity is the result of a measurement or calculation. A parameter is a letter representing a quantity. A formula is a relationship between parameters.

It remains to add a new term: — *the equation* .

The formula shows only the relationship between the parameters. For example, $d=2r$. But the formula cannot be used to calculate either the diameter or the radius. However, if we have managed to measure the radius, only then can we calculate the diameter. For example, let $r=7$. In this case, the formula becomes $d=2\cdot 7$. We have managed to equate the unknown parameter with a known value. This is what an equation is.

An equation is a formula that allows you to calculate the value of a parameter. This is done by equating the unknown parameter with a known value. In this case, this process is called finding the roots of the equation.

There is a tried-and-true method for finding the roots of an equation. What is it?

When equal quantities are subtracted from both sides of the equation, the equality of the sides is preserved. When both sides are divided by equal quantities—the

equality is preserved. When adding and multiplying—the same applies!

Solving an equation means using arithmetic operations to ensure that only the parameter, which needs to be calculated, remains on one side of the equation. That is, literally equating it with a known value.

For example, how do we solve the equation

$$x + 2 = 5$$

A child might say that this is easy and write the following:

$$x + 2 = 5$$

$$x = 5 - 2$$

$$x = 3$$

Everything is correct. But does he understand what he did, or did he just memorize the rule? Let's talk to him.

– I had a 2 on the left side of the equation. Where did it go?

– I moved it to the right side.

– What do you mean, moved? How many arithmetic operations do you know?

– Four. Addition, subtraction, multiplication, and division.

– There's no such thing as "moving." Moving is a physical action. How do you solve an equation using arithmetic operations, without moving anything? And why does the sign change when you move something?

We need to figure this out. First of all, let's say that equations originated as a description of the procedure of weighing on a scale.

Our equation describes the situation shown in the following figure:

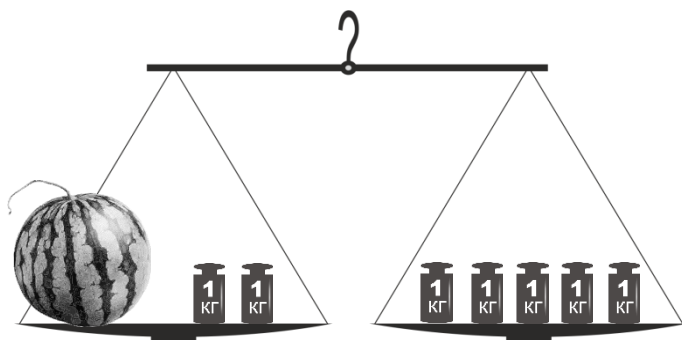


Рис. 8

What do you need to do physically to figure out how much the watermelon weighs? We need to make sure that only the watermelon remains on the left pan of the scale. Everything else must be removed. Let's remove two weights from the left pan. The balance will be disrupted. To restore it, we need to do the same with the right pan of the scale, i.e., remove two weights from the right side as well. The mass of the watermelon is 3 kg.

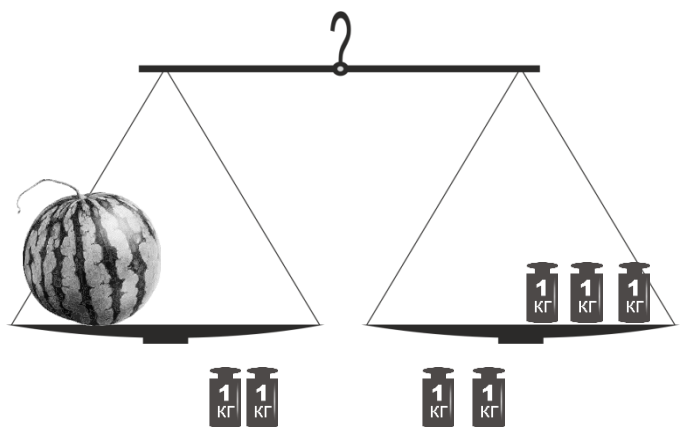


Рис. 9

Let's do the same with the equation. Here, too, we need to ensure that only the parameter we need to calculate remains on the left. We need to get rid of the term "2." We need to apply an arithmetic operation, not a physical one. We need to get rid of the 2 on the left using the inverse arithmetic operation. There was a "plus," so we need to put a "minus."

$$\begin{array}{r} x + 2 = 5 \\ - 2 \quad - 2 \\ \hline x = 3 \end{array}$$

– the initial situation on the scales
– we subtract 2 from both sides

Let's consider a more complex case. What is the mass of one apple if all four apples are exactly the same and the scales are in balance? (Рис. 10).

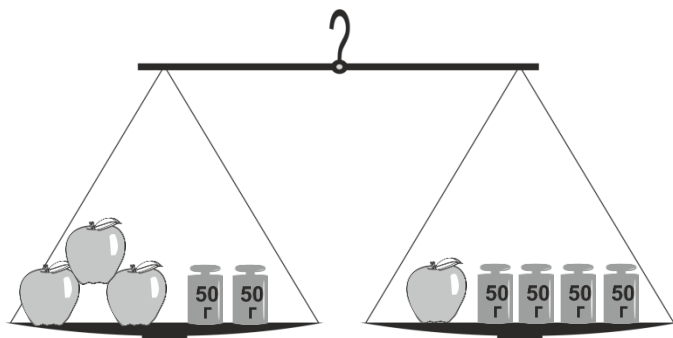


Рис. 10

Here, the mass of the apple is unknown, but the masses of the weights are known. How can we arrange things so that only the apple remains on one pan of the scale and only the weights on the other? Let your child think about it.

The apples must remain only on the left or only on the right. There are fewer on the right. Let's remove some apples from the right pan . The balance will be disrupted.

To restore it, we must remove exactly the same number of apples from the left. (Рис. 11).

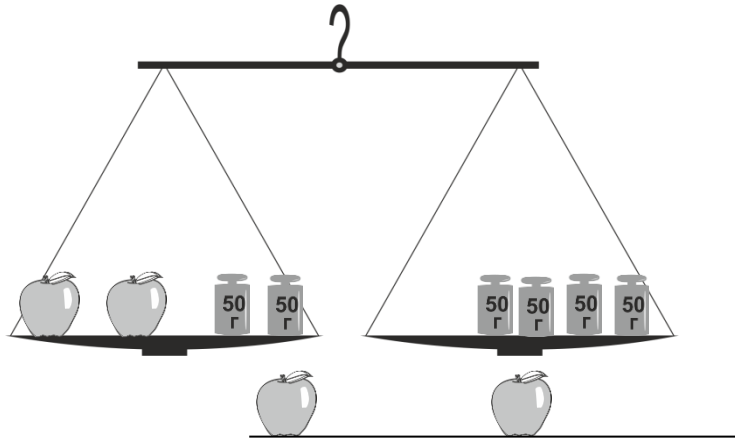


Рис. 11

Only apples should remain on the left pan. This means we need to remove weights from the left. Let's remove all 100 g. The balance will be disrupted again. To restore it, we need to remove 100 g from the right as well.

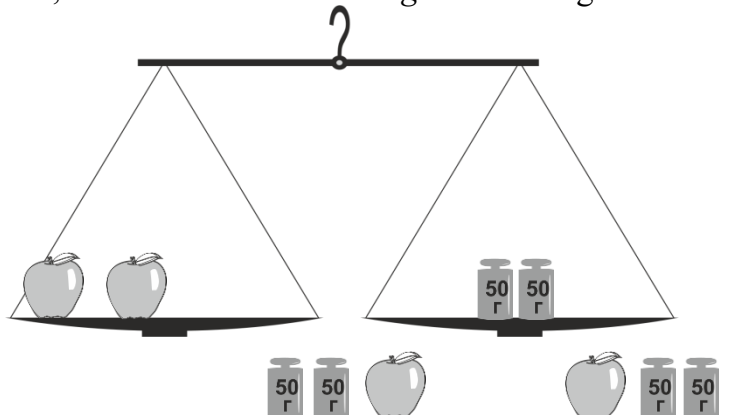


Рис. 12

There are two apples left on the left pan. We need to weigh just one. We could remove one apple from the left. But then we'd have to remove one apple from the right as well. But there are no apples there. And we don't know how much weight we need to remove from the right side to restore balance. After all, the mass of the apple is still unknown. But we know that we can divide both sides by two. It turns out that the mass of the apple is 50 g.

When solving the equation, the unknown x should remain on the left side, and the known number ... on the right.

"That's better," the child will say. "But how do we write down the solution to the equation correctly?"

It's simple! After all, equations originated as a way to record actions with scales.

Let's write down all our steps in order using the equation and find the mass of the apple.

$3x + 100 = x + 200$ – the initial situation on the scales

$$\begin{array}{r} -x \quad \quad -x \\ \hline 2x + 100 = 200 \end{array} \quad \text{– apples only on the left}$$

$$\begin{array}{r} -100 \quad -100 \\ \hline 2x = 100 \end{array} \quad \text{– only apples on the left}$$

$$2 = 2$$

$$x = 50 \text{ (g)} \quad \text{– only an apple on the left}$$

The equation above is relatively simple. It is given not so much to be solved as to demonstrate *the method* by which all equations are solved.

But what about transposition? In our schools, we use transposition to the left and right to solve equations. This

method is not suitable for solving equations, but rather for transforming formulas when we need to express one variable in terms of another. For example, consider the following formula:

$$a/b + x = d$$

We need to express the parameter a in terms of the other parameters. In this case, we subtract x from both sides and get:

$$a/b = d - x$$

This can be described as moving the parameter x from the left side of the formula to the right side with a change in sign—it was a plus, now it's a minus. Similarly, if we multiply both sides of the formula by d , we get:

$$a = (d - x) \cdot b$$

This can also be described as moving the parameter from left to right with a change in sign—it went from division to multiplication.

So, transposition can be used when a child is already good at solving equations and has moved on to working with algebraic expressions and formulas. This will be especially useful when deriving formulas and solving problems in physics class.

Nevertheless, it is important to show that the choice of method is up to the child.

22. How to solve problems?

In fact, a *method* is a powerful tool.

Is there a universal method for solving computational problems? Yes, there is. And it works for solving problems in geometry, mathematics, and physics...

The child is wary. He really needs this method. But he doesn't yet believe that such a method even exists. It sounds too good to be true!

Let's solve the previous problem another way. Let's review the problem.

"One ticket window sold 30 fewer tickets than the other. How many tickets were sold at each window if a total of 480 tickets were sold?"

Let's denote the parameter *a* as the number of tickets sold at the first ticket booth.

Let's denote *b* by the parameter the number of tickets sold at the second ticket window

Let *c* denote the number of tickets sold at both ticket windows.

List all the parameters appearing in the problem in the first column: *a*, *b*, *c*.

In the second column, let's write the relationships between these parameters, i.e., the formulas:

According to the problem:

$$a + b = c;$$

$$b = a + 30.$$

In the third column, using the relationship between these formulas, we write the equation. We substitute the

value of parameter b from the second formula into the first:

$$a + (a + 30) = c$$

Let's calculate a :

$$2a + 30 = c$$

$$\begin{array}{r} -30 \quad -30 \\ \hline 2a = c - 30 \\ \hline 2 \quad 2 \end{array}$$

$$a = (c-30)/2$$

Thus, the problem is solved parametrically, since in the last expression the only unknown is the parameter a , which needs to be calculated. Moreover, it now makes no difference how many tickets were sold in total. No matter how many were sold, we substitute their number for the parameter c and solve the equation :

In our case, $c = 480$.

Therefore,

$$a = (480-30)/2 = 225 \text{ (tickets);}$$

$$b = a + 30 = 225 + 30 = 255 \text{ (tickets).}$$

The problem is solved.

However, the word “parametrically” is a bit confusing. We need to clarify it.

Let's recall our very first formula. It was the formula for calculating the circumference $c = \pi d$. Give the child an old CD. He has already calculated its circumference. Have them calculate it again. They will measure the diameter ($d = 12$ cm), multiply the result by 3.14, and get 37.68 cm.

Now have them calculate the circumference of the inner circle (its diameter is 4 cm). They'll do that too. They'll get 12.56 cm.

Now let him calculate how many times the length of the outer circle is greater than the length of the inner circle.

Clearly, he will divide 37.68 by 12.56 and get 3. Three times as much.

Now ask: how many measurements and how many calculations did he perform?

He took two measurements and performed three calculations.

But it would have been possible to take two measurements and perform just one calculation... To do this, he would have had to solve the problem parametrically first.

Let $c_1 = \pi d_1$ — the length of the outer circle, and $c_2 = \pi d_2$ is the length of the inner circle. To calculate how many times the length of the outer circle is greater than the length of the inner circle, we need to divide the first length by the second. Let's write this down using parameters.

$$\frac{c_1}{c_2} = \frac{\pi \cdot d_1}{\pi \cdot d_2} = \frac{d_1}{d_2}$$

This is the *parametric* solution to the problem.

On the left-hand side are the parameters to be calculated, and on the right-hand side are the known (measured) parameters. Only after obtaining the parametric solution do we proceed with the calculations.

In our case, all that remains is to divide the diameter of the outer circle by the diameter of the inner circle.

$$\frac{c_1}{c_2} = \frac{d_1}{d_2} = \frac{12}{4} = 3$$

The child is delighted! But he hides it carefully. Yes, he has received a tool. But is it universal? Is it suitable for all problems? It's hard to believe... Let's consider another problem from 7th grade.

(An old problem). *A man is sent from Moscow to Vologda and ordered to travel 40 versts every day. The next day, a second man was sent after him. He was ordered to travel 45 versts every day. How many days will it take for the second man to catch up with the first?*

Let's use the same method.

In the first column, we'll write down the parameters, i.e., the letter symbols for everything that can be measured or counted:

a_1 – number of days the first messenger is on the road

a_2 – number of days the second messenger is on the road

b_1 – distance traveled by the first;

b_2 – the distance traveled by the second messenger;

In the second column, we will write the relationships between the parameters, i.e., the formulas:

$$a_1 = a_2 + 1;$$

$$b_1 = b_2$$

$$b_1 = 40 \cdot a_1 ;$$

$$b_2 = 45 \cdot a_2$$

In the third column, we will write the relationships between the formulas, i.e., the equation .

The relationship between the last three formulas is obvious:

$$40 \cdot a_1 = 45 \cdot a_2 ,$$

$$\text{but } a_1 = a_2 + 1.$$

Therefore,

$$40 \cdot (a_2 + 1) = 45 \cdot a_2$$

We have obtained the equation , in which there is only one parameter, . Let's evaluate it. First, let's expand the parentheses.

$$40 \cdot a_2 + 40 = 45 \cdot a_2$$

$$40 \times a_2 - 45 \times a_2 = -40$$

$$-5 \cdot a_2 = -40$$

$$a_2 = 8 \text{ (days)}$$

The problem is solved.

And what does it mean to solve a problem?

In our case, it means learning to use a tool called mathematics.

In fact, when performing any task, a person does not do it literally by themselves; they use tools. But a tool must be made, and one must learn how to use it.

We already have a tool called mathematics. But how do we use it—that is, what is the method?

Let's look at the solution to our problem again, describe what we did, and derive the definition of .

The solution is written as follows:

parameters	formulas	equations
a_1	$a_1 = a_2 + 1;$	$40 \cdot a_1 = 45 \cdot a_2 ;$
$a_2 - ?$	$b_1 = b_2 ;$	$40 \cdot (a_2 + 1) = 45 \cdot a_2$
b_1	$b_{(1)} = 40 \cdot a_1 ;$	
b_2	$b_2 = 45 \cdot a_2$	

calculations

$$40 \cdot a_2 + 40 = 45 \cdot a_2$$

$$40 \cdot a_2 - 45 \cdot a_2 = -40$$

$$-5 \cdot a_2 = -40$$

$$a_2 = 8 \text{ (days)}$$

At first glance, the diagram seems unusual.

Of course, one could say that the rate of convergence is $45 - 40 = 5$ (versts per day). The first traveler set out a day earlier and managed to cover 40 versts. Then, for the second traveler to catch up with the first at a rate of 5 versts per day, it would take $40/5 = 8$ (days).

There may be several methods of solution. We need to identify a universal method.

Let's consider another problem.

A boat traveled downstream for $2\frac{1}{3}$. It took 28 minutes longer to travel back upstream. Find the speed of the river current if the boat's speed in still water is 16.5 km/h.

Parameters	Formulas	Equations
$t_1 = 2\frac{1}{3} h.$	$S_1 = v_1 t_1$	$v_1 t_1 = v_2 t_2$
	$S_2 = v_2 t_2$	
$t_2 = ?$	$S_1 = S_2$	$(v_1 + v_r)t_1 = (v_1 - v_r)t_2$
$v_1 = 16.5$	$t_2 = t_1 + 28$	
$\frac{\kappa_M}{u}$	<i>min.</i>	$(v_1 + v_r)t_1 = (v_1 - v_r) (t_1 + \frac{7}{15})$
$v_r = ?$	$v_{l1} = v_1 + v_r$	
$v_{l1} = ?$	$v_{left2} = v_{left} - v_r$	
$v_{l2} = ?$	v_{right}	
$S_1 = ?$	$28 \text{ min} = \frac{28}{60} h$	
$S_2 = ?$	$\frac{28}{60} h = \frac{7}{15} h$	

calculations

$$(16.5 \frac{\kappa_M}{u} + v_p) 2\frac{1}{3} h = (16.5 \frac{\kappa_M}{u} - v_p) (2\frac{1}{3} h + \frac{7}{15} h)$$

$$(16.5 + v_{(p)}) \frac{7}{3} = (16.5 - v_p) \frac{14}{5}$$

$$38.5 + \frac{7}{3} v_p = 46.2 - \frac{14}{5} v_p$$

$$\frac{7}{3} v_p + \frac{14}{5} v_p = 46.2 - 38.5$$

$$\frac{77}{15} v_p = 7.7 \Rightarrow v_p = 1.5 \left(\frac{\kappa_M}{u} \right)$$

The solution seems cumbersome. But don't let that bother you. Step by step, we teach the child to identify *quantities* in the problem text—that is, everything that can be counted or measured—to label them *with variables*,

separate them from *the coefficients* that show relationships between *the variables*, construct *formulas* based on them, and set up *equations*. This is very important!

Over time, the child will do this faster, and the written form of the problem and its solution will become shorter and shorter. Moreover, after a while, the child will come up with a shorter version on their own.

Conciseness will be achieved by writing down only those parameters whose numerical values are known. There is no need to write down all parameters.

The same applies when solving equations. As they gain experience, children will shorten their solution notations. But they must arrive at the abbreviated notation on their own. For example, here is a concise notation for the solution to the last problem:

Given:	Solution
$t_1 = 2\frac{1}{3} h.$	$S_1 = (v_1 + v_r) t_1$ $S_2 = (v_1 - v_r) t_2$
$v_1 = 16.5 \frac{KM}{h}$	$S_1 = S_2 ; t_2 = t_1 + \frac{7}{15} h$
$v_r = ?$	$(v_1 + v_r) t_1 = (v_1 - v_r) (t_1 + \frac{7}{15})$ $(16.5 + v_r) \frac{7}{3} = (16.5 - v_r) \frac{14}{5}$ $\frac{77}{15} v_r = 7.7$ $v_r = 1.5 (\frac{KM}{h})$

Answer: The river's speed is 1.5 km/h.

To prepare for solving more complex problems, have the child repeat the following question after you: “If we know how much there is per unit, how do we calculate how much there is for a certain number of units?”

Once the child can repeat the question without hesitation, have them try to answer it. You’ll be surprised at how difficult it is for them to do so.

The answer should be: “To calculate how much there is per a certain number of units, you need to multiply the amount per unit by the number of units.”

Now the reverse question: “If we know how much there is per a certain number of units, how do we calculate how much there is per unit?”

Answer: “To calculate how much there is per unit, you need to divide the amount that corresponds to that number of units by the number of units.”

Why was all this necessary?

You will see later that the solution to practically all school problems in algebra, physics, and chemistry will boil down to answers to these two questions.

It will take quite a few hours to truly understand the meaning of the questions and answers and to be able to reproduce them freely. Be prepared for this, and don’t get angry or frustrated if the children don’t get it right away. Turn it all into a fun game.

So, the universal plan for solving the problem was as follows:

1. We translated the problem statement, written in Russian, into the language of mathematics using parameters and formulas.

2. From the formulas, we derived an equation. Solving the equation gave us the desired result in the language of mathematics.

3. We translated the result from the language of mathematics back into Russian.

23. What is the concentration of a solution?

In the morning, the child is having breakfast. If no one is in a hurry, you can gently bring up the topic of solutions and concentrations.

– You put two spoons of sugar in your cup of tea. What’s in the cup now?

– Tea and two spoons of sugar!

– We’ve made a sugar solution. The tea has become sweet. If you drink half a cup of tea now, will the “sweetness” of the remaining tea change?

– No, of course not!

– What if you drink half of that half?

– It won’t change either!

The amount of tea in the cup changes. The amount of sugar in the cup changes too, but the “sweetness” doesn’t change. How is that possible? How can we isolate this constant component responsible for the “sweetness”?

What percentage of the total amount of sweet tea is made up of the sugar dissolved in it? And why percentages? A percentage is one hundredth. So, we need to calculate how many hundredths of the total tea is sugar. Why hundredths?

The mass of sugar, compared to the total mass of tea in the cup, is very small. So, for convenience—to avoid dealing with fractions and parts—we decided to use hundredths, i.e., percentages. It is considered more convenient to say “a 5% sugar solution” than to say “five hundredths of a gram of sugar per gram of solution.”

– So, what fraction of the total tea in the cup is sugar? Can you calculate it?

– If we know how much sugar and how much tea there was, then we can calculate it.

– Let's say there were 10 g of sugar, and 200 g of tea and sugar combined.

– 10 divided by 200 is... One-twentieth, or zero point five...

– Five hundredths, i.e., five percent.

– So, the sugar makes up five percent of the tea!

– Of the sweet tea. Let's call the tea without sugar the solvent. Sugar is the solute. And all together, it's a solution. How do we now describe how much solute is in the solution?

– The solute makes up five percent of the solution's mass.

– And how do you say that without using the word "percent"?

– The solute makes up five hundredths of the mass of the solution.

– And what does that tell us? What do we know if we know that the solute makes up five percent of the solution?

– ?

– You divided the amount of dissolved sugar by the amount of solution and found how much sugar there is per unit of solution. In our case, there are five hundredths of a gram of sugar per gram of solution.

– So, in two hundred grams, there will be two hundred times as much?

– Exactly right. How much will that be?

– Two hundred multiplied by five hundredths is ten... Ten grams of sugar!

– Correct! That’s exactly how many grams of sugar were put in the cup.

Now we need to assign a term to this number of parts of the dissolved substance per unit of solution.

The amount of solute per unit of solution is called *the concentration of the solution*.

For example, a 12% hydrochloric acid solution means that 1 g of hydrochloric acid solution contains 0.12 g of hydrochloric acid.

The concentration is usually denoted by the parameter c .

Solution concentration is a specific quantity. It is similar, for example, to density or velocity. Knowing how many grams of solute are contained in 1 g of solution, you can always calculate how many grams are contained in any given amount of solution.

Let’s try to solve a problem.

Seven liters of water were added to a container containing 5 liters of a 12% aqueous solution of hydrochloric acid. What is the concentration of the resulting solution in percent?

First of all, let’s figure out what we need to calculate. We need to calculate how many liters of hydrochloric acid are contained in one liter of the resulting solution. This is the concentration of the resulting solution.

We proceed as usual.

Let m be the amount of solute;

c is the concentration of the solution;

M is the volume of the solution.

These parameters are related by the basic formula:

$$m = c \cdot M$$

We have two solutions—one before adding water and one after. Let's write down the basic formula for each of them.

$$m_1 = c_1 \cdot M_1$$

$$m_2 = c_2 \cdot M_2$$

Now, let's write down the relationship between the corresponding parameters based on the problem's conditions.

The relationship between M_1 and M_2 is simple—the volume of the second solution is 7 liters greater than that of the first. Therefore,

$$M_2 = M_1 + 7$$

The relationship between m_1 and m_2 is also simple—after adding water, the amount of acid in the solution will not change. Therefore,

$$m_1 = m_2$$

Using the last formula, let's combine all the parameters into a single equation.

$$m_1 = m_2 \Rightarrow c_1 \cdot M_1 = c_2 \cdot M_2 \Rightarrow$$

$$c_1 \cdot M_1 = c_2 \cdot (M_1 + 7).$$

And finally, the calculation. Let's substitute the numerical values. Just don't forget to convert the percentages to decimals.

$$12\% = 0.12;$$

$$0.12 \cdot 5 = c_2 \cdot 12;$$

$$c_2 = 0.6:12 = 0.05.$$

That comes out to five hundredths. Five hundredths of what? Five hundredths of a gram of acid in one gram of its aqueous solution.

Now we'll simply write the decimal fraction as a percentage

$$c_2 = 0.05 = 5\%$$

Answer. The concentration of the resulting solution is 5%.

Thus, if the original *1-liter* aqueous solution of hydrochloric acid contained 0.12 *liters* of acid, then after adding 7 *liters* of water, the new solution contains 0.05 *liters* of hydrochloric acid per liter.

24. How to measure angles?

We have learned to observe, describe what we observe, formulate definitions, assign terms to them, distinguish between groups, count, and measure. Moreover, we count not only whole groups but also their fractions. If we cannot measure or count something, we have learned to calculate it using a formula. The formula allows us to calculate what is difficult to measure using what is easy to measure. We are developing a universal algorithm of actions. Let's check if it works on something else.

A point has no definition, since it has no size and cannot be observed, and therefore cannot be described. No description means no definition. But the term “ ” — a point — does exist. It is a limiting term. Other definitions are constructed using it.

A line is also unobservable, and it too has no definition. But one can attempt to describe it using the term “point”: *a line* is a set of points arranged side by side. Although, if it is impossible to imagine a single point, it is difficult to imagine a set of points side by side... But what can one do—mathematics is an abstract science.

A line segment is a part of a line bounded on both sides by points.

A line limited on one side by a point is called *a ray*. A ray has a beginning but no end.

Two rays emanating from a single point form *an angle* .

ray

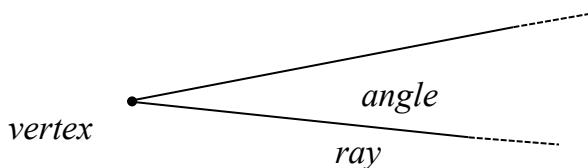


Рис. 13

The point from which the two rays of an angle originate is *the vertex* of the angle.

But angles come in different sizes. Therefore, they can and should be measured.

With what and how?

To measure means to compare with a standard.

The standard angle is the angle of 1 degree .

An angle of 1 degree is the angle formed by two radii that rest on $1/360$ th of a circle...

The vertex of the angle formed by two radii coincides with the center of the circle. Therefore, we will call the angle formed by two radii a *central angle* .

Consequently, the angle of *1 degree* is *the central angle* , *subtended by $1/360$ th of a circle*.

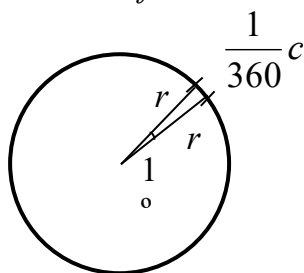


Рис. 14

Now let your child ask a question.

If they ask why $1/360$ th of a circle was chosen, and not $1/300$ th or $1/100$ th, reward them with a chocolate bar.

Indeed, why one three-hundred-and-sixtieth of a circle?

Oh, it's an old story! It began way back in Ancient Babylon. Today, any third-grader can handle numbers. But in those distant times, only the initiated knew how to use numbers. And those were usually monks in temples. One of the main deities was the Sun.

In the morning, the sun is in the east. In the evening, it sets. And at noon? What is the name of the highest point the sun reaches during the day?

Some will be surprised to learn that this point is called the *zenith*. At noon, the sun is at *the zenith*. This is the highest point. Then, at midnight, the sun will sink to its lowest point.

And what is the lowest point called?

It is called the *nadir*.

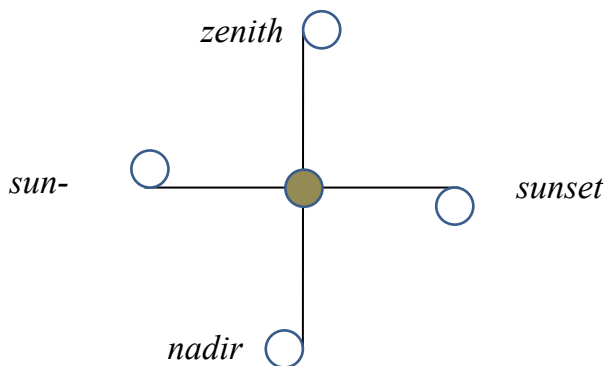


Рис. 15

The result is an ancient sacred symbol of the sun.

And how many sun disks fit along its daily path?

Ask a child if it's possible to measure the apparent diameter of the sun. Most likely, they'll say no. Have they

ever seen a solar eclipse, and if so, how did they observe it? Probably through darkened glass. So, is it possible to measure the diameter of the bright sun's disk on dark glass? Yes, it is.

This is the *apparent* (not actual) diameter of the sun. If we divide the sun's daily path by its apparent diameter, we get the number of suns that would fit on... But it is impossible to measure the sun's apparent path!

Let's take a different approach. What is the average time it takes for the sun to complete its daily path?

That's right, in 12 hours.

If we knew how long it takes the sun to travel a distance equal to its own diameter, we could divide 12 hours by that time and get the number of suns...

But how do you measure the time it takes for the sun to travel a distance across the sky equal to its own diameter? Let the child think about it.

It's easier for those who live by the sea. You can go out to the beach early in the morning, before the sun has risen. Everyone has a phone these days, and it has a stopwatch. Start the stopwatch when the top of the sun appears above the water on the horizon, and stop it when the sun has completely risen out of the water. How long did it take? Two minutes! The child will be surprised.

Let's divide 12 hours by two minutes.

Of course, first we'll convert 12 hours into minutes.

That comes to 720 minutes.

Divide that by two minutes, and you get 360 suns!

Three hundred sixty is the great magic number—s of the sun. It is divisible by 1, 2, 3, 4, 5, 6, 8, 9, and 10.

It is not divisible by 7.

But why?

In ancient times, it was believed that the number 360 is not divisible by 7 because 360 is the number of the sun, and 7 is the number of the moon... And the moon and the sun are eternal enemies.

But why is 7 the number of the moon?

And why are there seven days in a week?

Maybe the week is connected to the moon, too?

The child is already extremely eager to get the answers.

The moon orbits the Earth. It completes one orbit in... a month.

We see the moon because it is illuminated by the sun.

Since the moon has no atmosphere, the boundary between its illuminated and unilluminated parts is very distinct. As it orbits the Earth, the shape of the illuminated part changes for the observer. This is what is used to measure time.

Let's begin our observation when a round (full) moon is shining brightly in the night sky. This is a full moon. (Рис. 16).

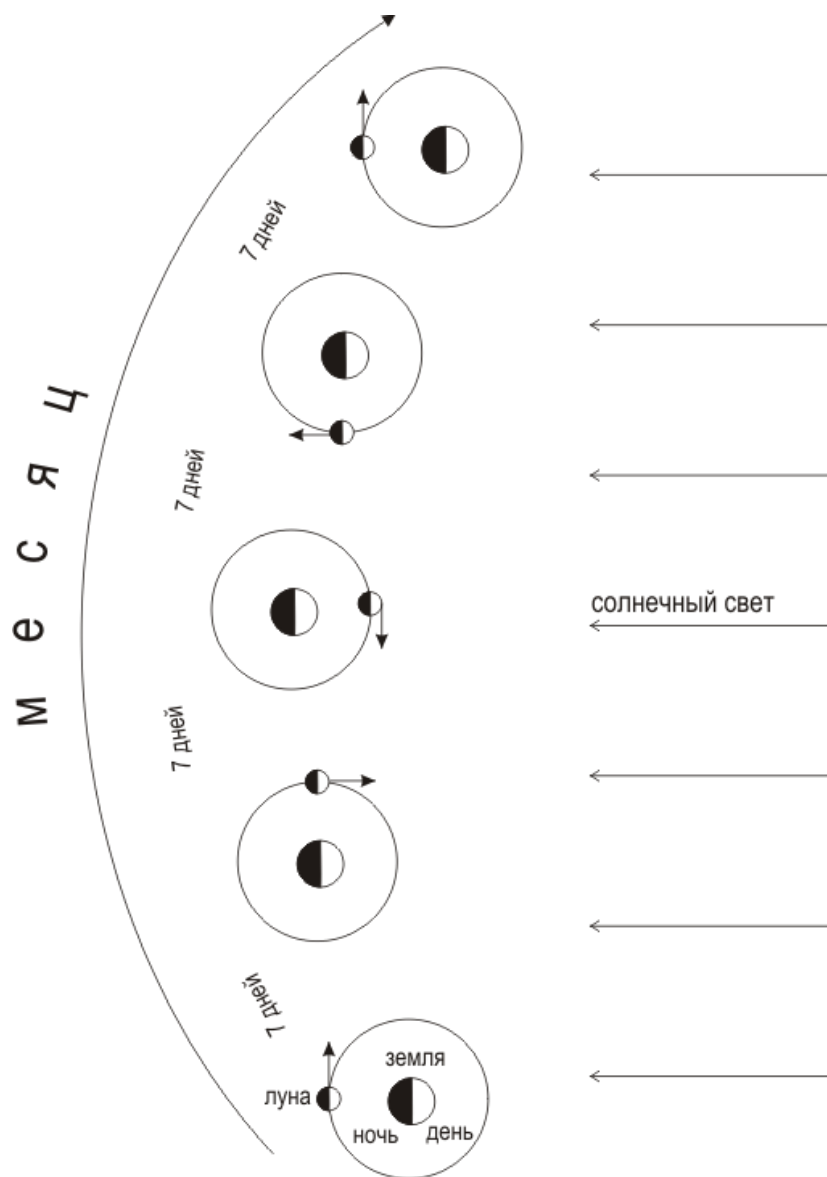


Рис. 16

After a few nights, we will notice that the shape of the moon's disk has begun to change. It will first turn into a crescent, then into a sliver... And then it will disappear altogether. The new moon arrives. Then the sliver of the moon will reappear; it begins to widen, turns into a crescent, and then becomes round again.

There is one month between full moons. And between the clearly distinguishable full moon, crescent, new moon, crescent, and full moon—there are seven days each.

That is why the number 7 is the number of the moon. And that is why there are seven days in a week.

The clearly distinguishable shapes of the moon are called lunar phases.

The time it takes for the Earth to complete one revolution around the Sun was called a great circle (1 year). The great circle was divided into twelve parts. Into 12, because 12 is a dozen, and everyone has that at their fingertips. 12 is the largest number you can count on the fingers of one hand. How so?

Ask your child to count to 5 on the fingers of one hand. That's easy.

Now, have them count to 12. Tell them that you saw how Grandma counted: use your thumb to count the joints on the other four fingers. There are exactly 12 of them. That's *a dozen*.

The Sun makes twelve stops at the twelve signs of the zodiac as it travels through the great circle. The twelfth part of the great circle becomes a month of the year. Exactly 360 discs of the Sun fit along its daily path, and 360 double Suns along its entire daily path, forming a small circle. Dividing the small circle, just like the large

one, into 12, we get 30 double Suns, which will be the “month” for the day. Dividing this “month” by 30, we get one double minute—the time it takes for the sun to travel a distance equal to its diameter. And two rays, based on the double sun, will give exactly 1 degree—a measure of angle—and there are 360 of them on the entire small circle. And 360 is the great magical number, which is the smallest of those divisible by 1, 2, 3, 4, 5, 6, 8, 9, and 10—that is, by the maximum number of numbers in the tens—the number of fingers on two hands.

360 is divisible only by the number of the moon, by the unruly number 7, which takes its name from the god Set, who killed his brother Osiris and, having dismembered his body into 72 pieces, scattered them across the world. And if you divide 360 by 72, you get 5. After all, that is exactly how much you need to add to 360 to get 365—the number of days in a solar year—because for exactly one year, the inconsolable Isis, wife of Osiris, gathered her husband’s remains...¹

Although, I think the explanation is much simpler. Dividing anything by 12 yields a number of parts that group into sets of 1, 2, 3, and 4—the maximum number of groups in a five (the fingers on one hand).

Dividing anything into 360 parts allows the parts to be grouped into sets of 1, 2, 3, 4, 5, 6, 8, 9, and 10—the maximum number of groups in a ten (the fingers on both hands).

Today, it’s simple. But back then...

* * *

¹ Thomas Mann. *Joseph and His Brothers*.

Twelve hours—day, twelve—night. Twelve double hours—a day. An hour has 60 minutes. A minute has 60 seconds. In total, an hour has 3,600 seconds. Ordinary clocks continue to tell us this ancient story. Teach your child to listen to this magic of numbers. Let them unite in their heart the present time and the ancient time. Only then will they find themselves, and they will not be just a grain of sand blown aimlessly by the wind. Only then will you be able to say that they are you, and they will carry on time and carry on life.

25. What is an angle of 1 radian?

A Story. Last time, you told the children a story about the number 360, the sun and the moon, the year and the month, Osiris and Set...

A story is a description of an event you observed. We already know that a right angle is 90 degrees. And an angle of one degree is a central angle that spans $1/360$ th of a circle. A central angle is an angle formed by two radii...

An angle, which is less than a straight angle, is called
an acute angle.

An angle that is greater than a right angle is called
obtuse.

If the two rays of an angle form a straight line, i.e., the angle is equal to 180 degrees, then such an angle is called *a full angle.*

Now we can measure angles. The tool for measuring angles is *a protractor*.

Have your child try making a protractor on their own. It's not that difficult. You need to cut out a semicircle, the base of which will, of course, be the diameter. To measure an angle, you need to align the center of the circle (the point in the middle of the diameter) with the vertex of the angle and position the semicircle so that its diameter coincides with one of the rays. The other ray will indicate the number of degrees on the semicircle... However, you must first mark these degrees.

How do you mark degrees on a semicircle?

Half of a circle contains 180° .

A small circle represents a degree. It's a little sun. We talked about its connection to the sun last time. Half a semicircle is 90° . Let's divide 90° into three parts— 30° each. Divide each 30° into three parts as well—that gives us 10° each. Halve those, and you get 5° each, and so on. This procedure is called calibration. The resulting tool is *a protractor*.

So, we can measure angles by comparing them to a standard, the one-degree angle (1°), which is the central angle, based on $1/360$ of a circle, and a central angle is defined as, formed by two radii, while a radius is a line segment connecting a point on the circle to its center, and the tool for measuring angles is a protractor, and...

It might seem a bit tedious, constantly repeating the same thing. But that's only how it seems to adults, for whom learning has become difficult. For children, on the contrary, it's very interesting... and important. Why is it important? Because they finally have something to say, and they can do it! They're able to form coherent speech, a story. And the story itself helps keep going.

Every new word (the term “ ”) allows them to repeat everything that came before. It's like in the fairy tale about the turnip: a new character appears, and you have to repeat everything that happened earlier: the beetle for the granddaughter, the granddaughter for the grandmother, the grandmother for the grandfather... Here there is repetition, reinforcement, logic, and consistency.

We have described the angles and measured them, but we do not yet have a formula.

Suppose we need to describe the length of the arc on which two radii rest. (Рис. 17).

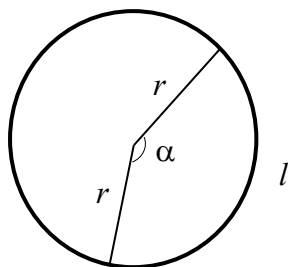


Рис. 17

The angle α can be measured with a protractor. The length of the radius r can be measured with a ruler. The length of the arc l cannot be measured with a ruler. Let's try to calculate it.

– What fraction of the circumference's length does the 1-degree angle correspond to? ?

– One three-hundred-and-sixtieth part.

– And the angle of 2 degrees?

– Two three-hundred-and-sixtieth parts.

– And the angle of 7 degrees?

– By seven three-hundred-and-sixtieth parts.

– And the angle of α degrees?

– By α three-hundred-and-sixtieth parts.

– Try writing this down, if the fraction of the circle is the arc on which the angle α rests.

The result should be:

$$l = \alpha \cdot \frac{1}{360} \cdot c$$

Here, the angle α can be measured with a protractor. And the circumference cannot be measured with a ruler. But we can calculate the circumference using the formula πr .

Therefore,

$$l = \alpha \cdot \frac{1}{360} \cdot 2\pi r = \frac{\alpha \pi r}{180}.$$

We have obtained a formula for calculating the length of the arc subtended by the central angle .

Degrees are good for measurements... But they are inconvenient for calculations. The formulas become quite cumbersome. Therefore, another unit of angle measurement was introduced. This is the angle in 1 radian (1 *rad.*).

An angle of 1 radian is formed by two radii (r) resting on a circular arc (l) equal to the length of the radius.

Therefore, *the angle in radians is the central angle , subtended by an arc of a circle equal to the length of the radius.*

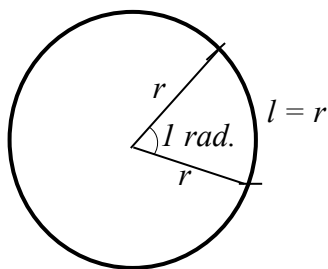


Рис. 18

If we divide the circumference by the radius, we get the number *of radians* in the circle, since each arc equal to the radius subtends an angle of one radian.

Recalling the formula for the circumference ($C = 2\pi r$), we get:

$$\frac{c}{r} \frac{2\pi r}{r} = 2\pi.$$

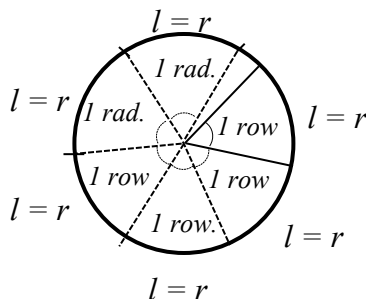


Рис. 19

So, the number of angles in 1 radian that fit into a circle is equal to 2π , or ≈ 6.28 . After all, we know that the number π is the number of diameters that fit along the circumference of a circle, and it is equal to ≈ 3.14 .

But there are 360 degrees in a circle.

Therefore, $2\pi \text{ rad.} = 360^\circ$.

Accordingly, 1 degree will be $2\pi/360$ radians or, after simplification,

$$1^\circ = \pi/180 \text{ rad.}$$

One could write 3.14 everywhere instead of π and get specific numbers, but in most cases this would result in a fraction, which is quite inconvenient to work with. For example, $30^\circ = \pi/6 \text{ rad.} = 3.14/6 \text{ rad.} = 0.523333... \text{ rad.}$

Considering that the number π itself is only rounded to 3.14, and in reality it is 3.1415926... followed by an infinite series of digits, it is better to leave the parameter π in the radian measure of the angle. It is easy to get used to the fact that $\pi/4 \text{ rad.}$ is 45° .

Going forward, the unit “degrees” (radian) is not used for angles expressed in radians that are multiples of

π . We will write and say that the angle is equal to, for example, half a pi ($\pi/2$) or one-sixth of a pi ($\pi/6$), implying that these are, of course, radians.

The arc length subtended by an angle of 1 radian is equal to the radius (r). Consequently, the arc length (l) subtended by an angle of 2 radians will be twice as long, 3 radians three times as long, and φ radians φ times as long; that is, we obtain the formula for calculating the arc length subtended by a central angle φ , measured in radians.

$$l = \varphi \cdot r$$

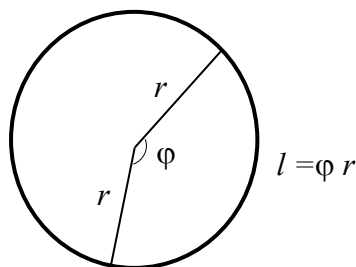


Рис. 20

Therefore, the length of the arc subtended by the central angle in π radians will be equal to $l = \pi r$.

Here, the length of the radius can be measured with a ruler. The angle can be measured with a protractor in degrees and converted to radians by multiplying the number of degrees by $\pi/180^\circ$. The product of the radius length and the angle measure will give the length of the arc subtended by this angle. It is difficult to measure the length of this arc with a ruler. We calculated it. The formula once again helped us calculate what is difficult to measure using what is easy to measure!

26. What are the criteria for triangles to be congruent?

We know that a closed curve can form a circle. But what if we construct a closed polygonal curve? A polygonal curve is a curve formed by a series of connected line segments. The simplest closed polygonal curve is formed by three connected line segments. This is a *triangle*. A triangle has three angles and three sides. The vertices of the angles are also called *the vertices of the triangle*. The bottom side is called *the base*. The vertices are denoted by uppercase Latin letters, and the lengths of the sides by lowercase Latin letters corresponding to the letters of the opposite vertices. The measures of the angles of a triangle are denoted by Greek letters. (Рис. 21).

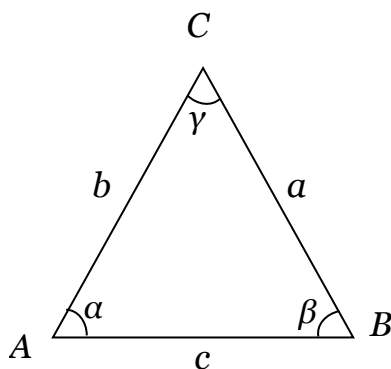


Рис. 21

Triangles are considered congruent if, when superimposed, all their sides and angles coincide.

But it is not necessary to compare all three sides and all three angles. It is sufficient to compare only some of them. Which ones?

The first criterion for the equality of triangles states: *if two sides and the angle between them of one triangle are respectively equal to two sides and the angle between them of another triangle, then such triangles are equal.*

The second criterion for the congruence of triangles states:

if a side and the two angles adjacent to it in one triangle are respectively equal to a side and the two angles adjacent to it in another triangle, then these triangles are congruent.

The third criterion for the equality of triangles states: *if the three sides of one triangle are respectively equal to the three sides of another triangle, then these triangles are congruent.*

Triangles may not be congruent but may be similar in shape, i.e., they may have equal angles but unequal sides. Such triangles are called similar.

Criteria for Similarity of Triangles

The first criterion for similarity of triangles states: *if two angles of one triangle are respectively equal to two angles of another, then these triangles are similar.*

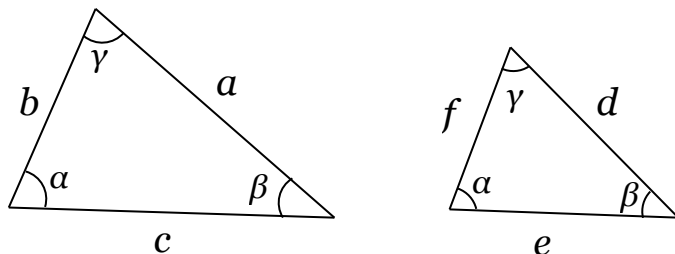
The second criterion for similarity of triangles states: *if three sides of one triangle are proportional to three sides of another triangle, then these triangles are similar.*

The third criterion for the similarity of triangles states: *if two sides of one triangle are proportional to two sides of another triangle, and the angles included between these sides are equal, then such triangles are similar.*

Which sides of similar triangles are called corresponding (or analogous)?

Those that lie opposite equal angles.

Consequently, for similar triangles, we can write a formula that relates the lengths of their sides.



$$\frac{a}{d} = \frac{b}{f} = \frac{c}{e}$$

If the lengths of two sides of one triangle and the length of at least one side of a similar triangle are known, then the lengths of all sides of both triangles can be calculated.

Similarity of triangles is denoted by the symbol “ $\sim$.” For example, the notation $\Delta ABC \sim \Delta KLM$ means that these triangles are similar. The ratios of their corresponding sides are proportional (equal). But how can this be proven?

First, we need to consider the types of angles and the relationships between them.

27. Types of Angles

Let's continue our exploration of angles.

Opposite angles formed by intersecting lines are called *vertical angles*.

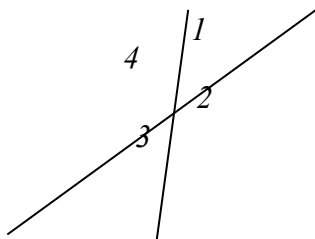


Рис. 22

Angles 1 and 3 are vertical. Angles 2 and 4 are also vertical.

Vertical angles are equal.

Angles that share one common ray, while the other two rays form a straight line, are called *adjacent angles*.

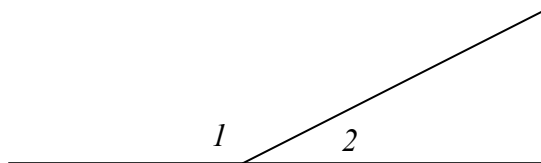


Рис. 23

Angles 1 and 2 are adjacent.

The sum of the adjacent angles () is 180° .

Parallel lines are non-intersecting lines lying in the same plane.

A line that intersects parallel lines is called a *secant*.

Two parallel lines and a secant form eight angles. Let's examine them.

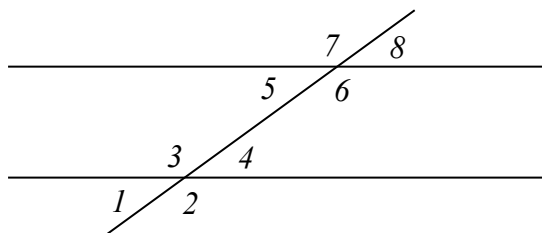


Рис. 24

Have the child point out the adjacent and vertical angles in the figure.

Here, 3, 4, 5, and 6 are *interior angles*. They are located inside (between) the parallel lines.

Angles 3 and 6 are *interior alternate angles*. They are located inside the parallel lines, alternating across the intersecting line. Interior alternate angles are equal.

Angles 4 and 5 are also interior opposite angles.

Angles 4 and 8 are *corresponding angles*. Corresponding angles are equal.

Now we can formulate and prove the first theorem.

28. How do we prove Thales' theorem?

's Theorem states: *if two parallel lines, which are angle bisectors, divide one side of the angle into equal segments, then they also divide the other side into equal segments.*

Let's draw a diagram. Let's demonstrate that the general rule applies here as well: I observe, I describe what I observe...

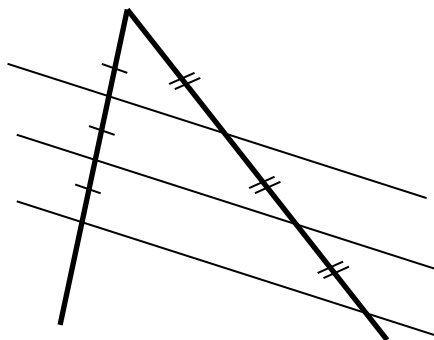


Рис. 25

Have the child repeat the conditions of the theorem, demonstrating each point on the diagram. Here are the parallel lines; here they intersect the angle's rays and divide one ray into equal segments; here they divide the second ray of the angle into equal segments... Are they equal? We need to prove it!

Next comes an important point. The child must learn to use the language of mathematics, and this requires practice with the language. Place a point (A) as shown in рис. 26 .

Have the child describe the position of this point. For them, this will be an unsolvable problem. Don't keep them

waiting for too long; describe it yourself: it is *the point of intersection of one of the parallel lines and the second angle ray*. Make sure they repeat this phrase several times.

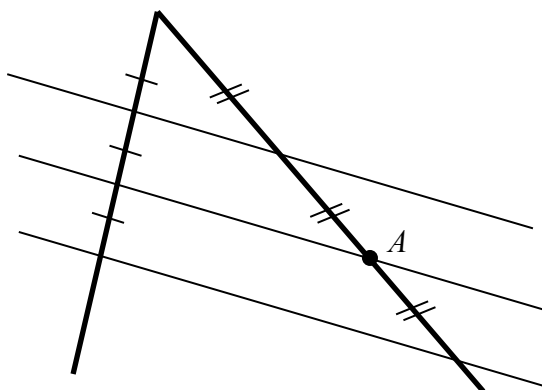


Рис. 26

Now we can present the text of the proof.

To prove Thales's theorem, draw a line parallel to the first ray through the point of intersection of one of the parallel lines and the second angle ray.

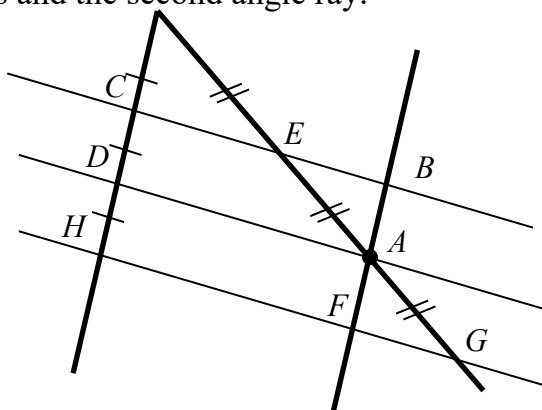


Рис. 27

We obtain two triangles, AEB and AFG . If we prove that these triangles are congruent, this will mean

that the line segments EA and AG are also equal. Which is what we need to prove.

Let's compare the angles in these triangles. Have the child identify the vertical and opposite angles. Help them correctly identify the parallel lines and the bisector.

It turns out that all three angles of the first triangle are equal to all the angles of the second. Is this enough to conclude that the triangles are congruent? The child will firmly state that it is enough.

In fact, of course, it is not.

Draw a triangle and draw a line that intersects its sides parallel to the base.

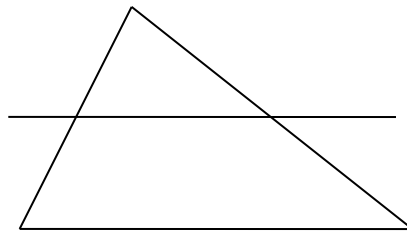


Рис. 28

We get two triangles: a large one and a small one. All three angles of the large triangle are equal to the three angles of the small one, respectively... But the triangles themselves are not congruent. They are only similar.

For triangles to be congruent, it is also necessary that at least one side of one triangle be equal to a side of the other triangle.

Consider quadrilateral ABCD atрис. 27 .

Sides AB and CD are parallel by construction. And sides BC and DA are parallel by the condition of the theorem. Therefore, quadrilateral ABCD is a parallelogram.

In a parallelogram, opposite sides are not only parallel but also equal. Therefore, segment AB is equal to segment CD.

But the length of segment CD is equal to DH by the theorem's condition. In turn, DH is equal to AF, since the quadrilateral ADHF is also a parallelogram. But this means that segment AB is equal to segment AF. And this means that triangles AEB and AFG are congruent.

In congruent triangles, equal sides lie opposite equal angles.

Angles EBA and AFG are equal as corresponding interior angles. Therefore, the segments EA and AG opposite them are also equal.

's Theorem is proven.

* * *

Let the child not yet know much; let him not remember all the criteria for the equality of triangles and be unable to prove that the opposite sides of a parallelogram are equal in length. That doesn't matter. What matters is that there is a sufficiently long and clear text in mathematical language that they can reproduce. And the child gradually gets used to the new language, begins to feel it.

29. The Theorem on the Median of a Triangle

What does Thales's theorem on the median of a triangle tell us? Where can it be applied?

Let's give the child a task. Suppose we are building a country house. The attic turned out to be very high. We decided to make a room in the attic as well. First, we need to connect the midpoints of the left and right roof slopes with a beam. You give this task to the child. Let them describe the sequence of their actions.

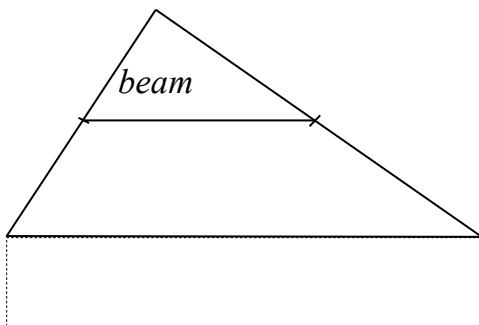


Рис. 29

After a few attempts, the result will be something like this... At the very least, make sure the text turns out something like this.

"I set up the ladder. I climb up to the roof, to the very ridge. I lower a tape measure along the left roof slope and measure its length. I divide the length in half. Risking my life, I climb down and mark the position of the slope's midpoint. Then I do the same on the right side. Then I climb back down. I climb the ladder into the attic. I find my marks and measure the distance between them with a

tape measure. Then I go down to the yard. I find a beam. I saw off the required length. I carry it up to the attic and nail it in..."

Now tell me, Thales's theorem will help eliminate the most dangerous part of the job. There's no need to risk your life climbing onto the roof.

Let's look at our diagram again. The roof resembles the diagram for Thales's theorem. Only in our case, the two horizontal segments (AB and CD) divide the left angle ray and the right angle ray into equal segments. Therefore, by Thales's theorem, these two segments are parallel.

Draw a line through the midpoint of the right side of triangle OCD that is parallel to its left side.

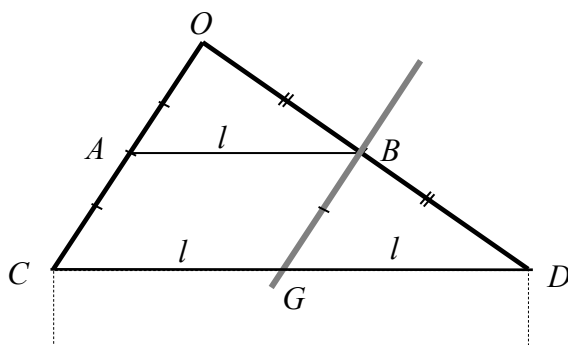


Рис. 30

Quadrilateral ABGC is a parallelogram, since side AC is parallel to side BG by construction, and side AB is parallel to side CG by Thales' theorem. Therefore, the length of segment AB is equal to the length of segment CG. Great!

Now let's consider the two triangles OAB and BGD.

By the given condition, segments OB and BD are equal.

Segment BG is equal to segment AC, which in turn is equal to segment OA (by the given conditions).

Angle AOV is equal to angle GBD (corresponding angles). Two sides and the angle between them in one triangle are equal to two sides and the angle between them in the other triangle. Therefore, these triangles are congruent.

In congruent triangles, equal sides lie opposite equal angles. Therefore, segment AB is equal to segment GD.

The line segment connecting the midpoints of the sides of a triangle is called the median of the triangle .

We have derived the theorem on the median of a triangle.

The median of a triangle is parallel to the base of the triangle and equal to half its length.

$$\text{That is, } AB = \frac{CD}{2} .$$

To calculate the length of the median of a triangle, it is enough to measure its base... and divide its length in half.

So, to complete our task, it was enough to measure the length of EF directly on the ground. Divide it by two. Cut a beam of that length. Lift it into the attic and, parallel to the floor, raise it until it touches the roof and nail it down. The beam will connect the midpoints of the roof slopes...

30. Theorem on the median of a trapezoid

Let's draw a trapezoid (ABCD). Let's construct a line segment within it connecting the midpoints of the sides—this will be the median of the trapezoid (EF). And let's consider whether it is possible to calculate its length by measuring only the bases of the trapezoid (AD and BC)?

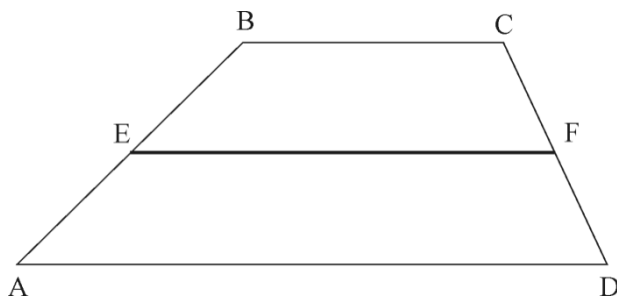


Рис. 31

At first, it may seem that there is no solution. But let's remember that a complex problem is often solved by breaking it down into simpler problems. We know how to calculate the length of the median of a triangle. Can't we also reduce our problem to a triangle with a median? Yes, we can.

Draw a line segment from the left vertex through the midpoint of the right side of the trapezoid to where it intersects with the extension of its base.

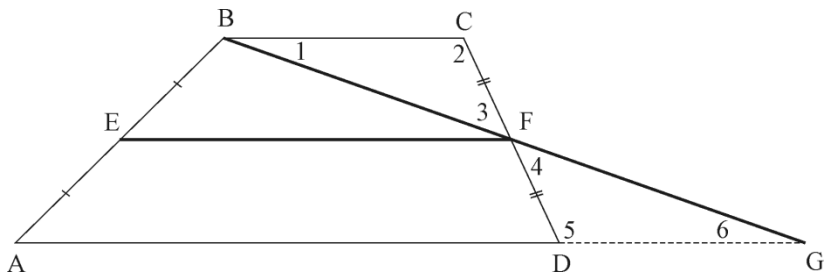


Рис. 32

We obtain a large triangle ABG and two small triangles BCF and FDG.

Let's prove that triangles BCF and FDG are congruent.

Here, angles 3 and 4 are equal, as they are vertical.

Angles 1 and 6 are equal, as they are opposite interior angles. Angles 2 and 5 are also equal.

Side CF is equal to side FD, by the given condition.

Therefore, triangles BCF and FDG are congruent.

This means that segments BF and FG are also equal.

Now let's consider the larger triangle ABG. In it, segment EF connects the midpoints of the triangle's sides, i.e., it is the median of triangle . It is parallel to the base, and its length, as we proved above, is equal to half the length of the base.

$$\text{That is, } EF = \frac{AG}{2}.$$

But $AG = AD + DG$, and $DG = BC$.

Therefore,

$$EF = \frac{AD + BC}{2}$$

The length of the median of a trapezoid is equal to half the sum of the lengths of its bases.

There can be several proofs of the same theorem. Let your child independently complete another version of the proof of the theorem on the median of a trapezoid.

Let's construct a trapezoid and draw its median. Now let's draw a diagonal of the trapezoid—connect two non-adjacent vertices with a line segment. This will result in two triangles.

Have your child draw these triangles. Note the bases of the triangles. They are the bases of the trapezoid. It remains to prove that the segments into which the median of the trapezoid is divided by the diagonal are the medians of the triangles.

The sum of their lengths is equal to the median of the trapezoid.

Now have the child write this down as equations and perform the addition.

This will give the formula for the median of a trapezoid.

31. Theorem on the sum of the angles of any triangle

We already have three proven theorems. Now we need to give the child the opportunity to prove the theorem themselves. Let them prove that the sum of the angles of a triangle is 180° . Let them construct a triangle. The relationships between the angles are not immediately apparent here. To establish these relationships, we need to make an additional construction. Let's draw a line through the vertex of the triangle that is parallel to the base.

Let's label the resulting angles. (Рис. 33). Now the relationships are clear.

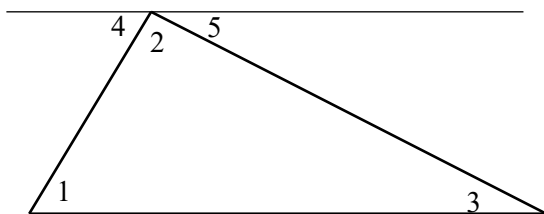


Рис. 33

– Angles 4 and 1 are alternate interior angles. They are equal. Angles 5 and 3 are also equal.

– What is the sum of the angles 4, 2, and 5 ()?

– 180° .

– Great! And what is the sum of angles 1, 2, and 3?

– ?

– In the first sum, replace angles 4 and 5 with angles 1 and 3, which are equal to them.

– It will also be 180° .

So, in any triangle, the sum of its angles is 180° .

And in a quadrilateral? Any quadrilateral can be divided into two triangles by drawing a diagonal. Therefore, the sum of the angles in a quadrilateral is 360° .

And in a pentagon?

Here, two diagonals will divide the pentagon into three triangles. This will give 540° .

In any polygon, the number of diagonals drawn from one of its vertices is two less than the number of its sides. Therefore, there will be the same number of triangles. Then, the sum of all the angles of the polygon can be calculated using the formula:

$$A = 180^\circ \cdot (n - 2)$$

Here, A is the sum of the angles; n is the number of sides of the polygon.

32. What is area?

Let's continue practicing descriptions and getting familiar with formulas—these great tools. But first, let's remember why we're learning these lessons.

We may not actually need these lessons on radii and formulas themselves... What we need are the skills that these lessons help develop. For example, if you bought your son dumbbells, that doesn't mean he needs dumbbells specifically. Rather, he needs physical development, which can be achieved with their help.

If your child cannot describe what can be directly observed, then all the more so he will not be able to describe what is difficult or impossible to observe directly. How will he describe thoughts, feelings, and experiences?... How will he describe what is on his mind if he cannot describe a sunrise, a summer rain, or the radius with a diameter ?

Let's continue studying the simple in the hope that it will help us understand and master the complex.

The child is still impressed by how the formula helped calculate the circumference and the arc length subtended by the central angle . Let's reinforce this impression.

Let's look at another example—measuring and calculating the area.

And what is the area of ?

For points and lines drawn on the board, a set of points located close together is *a straight line*...

A set of parallel lines is *a plane*.

A part of a plane bounded by a closed curve is *an area* .

The area of a plane is the same as a segment for a line.

To measure means to compare with a standard.

Therefore, to measure an area means to compare it with a standard of area.

And what do we have as a standard of area?

Of course, the area of a square with a side equal to 1 meter—one square meter.

Ask a child how big the classroom at their school is. They might say it's about three or four of their rooms... But more precisely?

We need a comparison to something everyone knows. That something is the area of a square with a side of 1 meter. After all, everyone knows what a meter is, so they can picture a square with a side equal to one meter.

But your child isn't that simple. What if they ask, "What is a square?"

"A square is a rectangle with all sides equal," you'll answer.

"And what is a rectangle? We haven't covered that yet!"

A rectangle is a quadrilateral with all right angles.

A rectangle with all sides equal is *a square*.

A square with a side length of 1 meter defines the area , which is taken as the standard unit of area and called "1 square meter."

To measure the area of means to compare it to the standard of area.

Clearly, cutting a square meter out of plywood and taking it to school to measure the area of the classroom is somewhat inconvenient. What should be done?

We need to replace the measurement with a calculation.

Suppose we need to measure the area of a rectangle drawn on a sheet of paper. Since the area is small, let the unit of measurement (standard) be a square with a side length of not 1 meter, but 1 centimeter. To measure the area, we need to make a standard unit, i.e., cut out a square with a side of 1 cm, and compare the desired area to it. But this, as already mentioned, is rather inconvenient.

If we measure the length of one side of the rectangle in centimeters with a ruler, we will find out how many square centimeters fit along that side.

Similarly, by measuring in centimeters the length of the other side of the rectangle that forms a right angle with the first side, we can determine the number of squares that fit along it.

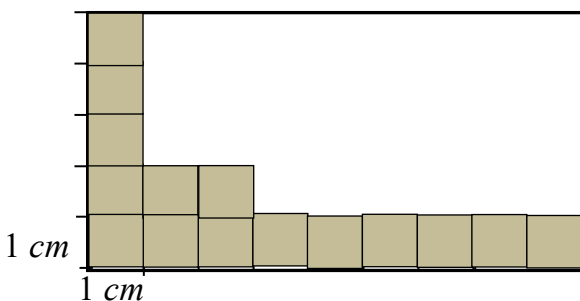


Рис. 34

The figure shows that there are nine columns of five squares each, i.e., a total of 45 squares.

The total number of squares is equal to the product of the lengths of the sides forming the right angle .

We no longer need to measure the area . We have calculated it. Let's write down the formula for calculating the area of a rectangle.

$$S = a \cdot b,$$

where the parameter S is the area of the rectangle , the parameter a is its length, and the parameter b is its width.

The letter S was chosen to denote area for a reason. In English, "*square*" means a square. Hence, S stands for the number of squares.

We haven't eliminated the need to measure altogether. But we've replaced the rather tedious calculation of the area of a rectangle with the fairly simple measurement of its length and width.

Just as in life, mastering new technologies allows us to solve rather complex everyday problems with minimal effort. But this is for those who understand and who have managed to master measurement and calculation technologies at least at a third-grade level.

33. How do you derive the formula for the area of a triangle?

Despair. Perhaps this word best describes the state of mind of some of my readers. They feel like they understand everything, but they also realize they won't be able to repeat or remember any of it. All these areas, squares, radii, π , and formulas lead to... Yes, yes—despair. Children feel the same way.

“I'll never, ever be able to do this!” they sometimes exclaim.

You will! Just don't give in to despair.

First of all, let's remember that it may well be that none of you will ever have to calculate the area of a rectangle. But we've already said that what we need isn't the calculations themselves, but the development of our abilities. We need to learn how to think. Those who believe that the ability to think is an innate human quality are mistaken. The only thing that is innate is the ability to learn how to think.

Think! How often do we say that to a child?

Think! That's interesting... I'd think about it if I knew how to do it.

Let's keep learning to think using simple models.

By comparing the size of the rectangle to that of a square with a side length of 1 cm, we can describe the area of the rectangle, i.e., *measure* it.

By multiplying the lengths of the sides forming a right angle, measured with a ruler in centimeters, we can *calculate* the area of the rectangle.

But try, with only a ruler in hand, to calculate or measure the area of the circle .

A circle is a plane bounded by a circle .

A circle ... Not a single straight line segment! What should I measure? How should I calculate it?

To start with, let's take a simpler problem: let's calculate the area of the triangle . It's practically impossible to measure it. The squares won't fit inside the triangle. So how do we calculate it? Which parameters should we measure, and which formula should we use?

Of course, you could have a cheat sheet with all the formulas on hand, plug in the data, and pretend you've passed the exam.

But, first of all, where do you get the data, and second, why pretend? It's clear that in most schools, the second law of socialism still applies: what matters isn't the "work" itself, but the report on the "work" done. But why do you need this, and to whom are you planning to submit the report?

Let's try to extract the data on our own and derive the necessary formulas. No one can take this skill away from you, and it will help you create formulas on your own to solve more complicated real-life problems.

Let's draw a diagonal in the rectangle. A diagonal is a line segment connecting two non-adjacent vertices of a rectangle. We get two triangles. They are exactly the same, congruent. Their areas are equal to each other. Therefore, the area of one triangle is equal to half the area of the entire rectangle. And we already know the formula for the area of a rectangle: $s = ab$.

$$\text{Therefore, } S_{\Delta} = \frac{a \cdot b}{2} .$$

To calculate the area of a right triangle, you need to measure the lengths of the two sides that form a right angle and divide their product by two...

Note: the solution turned out to be quite simple. But it only seems simple once it has been found. Similarly, a new problem may seem difficult to solve at first, but in the end, everything turns out to be quite simple. You have to learn not to be afraid and to act boldly. To act, even without fully imagining where that action will lead.

Let's continue.

What if the triangle isn't a right triangle?

The bottom side of the triangle is called *the base*. Let's denote its length by the parameter a .

Draw a line segment from the vertex of the triangle to its base such that a right angle is formed between this segment and the base.

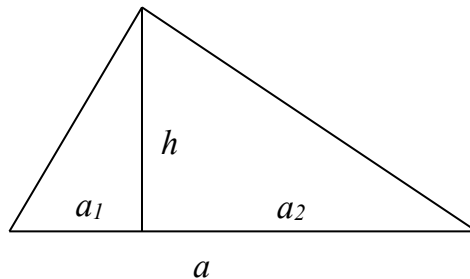


Рис. 35

We say, "Let's drop *a perpendicular*." A perpendicular is a line segment drawn to another line segment at a right angle.

A perpendicular dropped from the vertex of a triangle to its base is called *the height* of the triangle.

Let's denote its length by the parameter h . The height divides the base of the triangle into two parts. Let's denote their lengths as a_1 and a_2 , respectively.

By drawing the height, we have divided an arbitrary triangle into two right triangles. The sum of the areas of these two right triangles will give the desired area of the triangle. And we know how to calculate the areas of right triangles: we need to multiply the lengths of the sides forming a right angle and divide the result by two.

Let's write down the formulas for the areas of the two triangles:

$$S_{\Delta 1} = \frac{h \cdot a_1}{2} \text{ and } S_{\Delta 2} = \frac{h \cdot a_2}{2}.$$

Add these two areas together to get the desired area of an arbitrary triangle:

$$S_{\Delta 1} + S_{\Delta 2} = \frac{h \cdot a_1}{2} + \frac{h \cdot a_2}{2} = \frac{h \cdot a_1 + h \cdot a_2}{2} = \frac{h(a_1 + a_2)}{2}$$

But $(a_1 + a_2)$ is the length of the base a .

Therefore, the area of any triangle is equal to

$$S_{\Delta} = \frac{ah}{2}$$

This reads as follows: *the area of a triangle is equal to half the product of the lengths of its base and height.*

We have replaced the rather tedious measurement of the area of the triangle with a simple measurement of the lengths of its base and height and a calculation. What a time-saver!

But is it possible to derive the formula for the area of a triangle without algebra? Geometry makes this possible. Let's extend the triangle into a rectangle (Fig. 32).

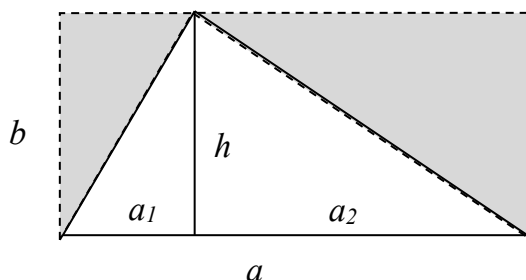


Рис. 36

The area of the resulting rectangle is equal to the product of the lengths of its sides forming a right angle , i.e., $S = ab$. The height of the triangle h is equal to the length of the side of the rectangle b , and it divides the triangle into two right triangles.

The area of the left right-angled triangle is equal to half the area of the left rectangle. The area of the right right-angled triangle is equal to half the area of the right rectangle. Then the area of the entire triangle is equal to half the area of the entire rectangle. That is, half of $S=ab$.

Consequently, substituting b with h , we have:

$$S_{\Delta} = \frac{a \cdot b}{2} = \frac{a \cdot h}{2}$$

In the same way, we can calculate the area of the trapezoid .

A trapezoid is a quadrilateral in which only two sides are parallel.

Suppose you need to paint the floor in your child's room. Let's say the floor in the room is shaped like a trapezoid. Painting 1 square meter costs, for example, 100 rubles. How much money do you need to paint the entire

floor? Let your child help you calculate the exact amount you'll need to pay the painter.

Clearly, you need to determine the floor area . How? Measure it? But that's quite a hassle. Cut out a square meter from plywood and compare...

We need to calculate it. But using what formula?

Let the child derive the formula themselves. They have everything they need to do so. Just hint that a trapezoid can be broken down into a rectangle and two right-angled triangles, the formulas for calculating the areas of which we already know.

34. How do you calculate the area of a trapezoid?

Calculation is the greatest invention. Calculations allow us to drastically reduce the time, effort, and resources needed to determine the necessary parameters. Calculations allow us to model a process and analyze future results with virtually no resource expenditure.

How can you learn this?

Keep studying.

Let's check the homework assigned in the previous lesson. Was your child able to derive the formula for calculating the area of a trapezoid? And how accurate is the formula they came up with: ? After all, to verify it, you need to compare the area calculated using the formula with the measured value. If the results match...

Is there a different, more efficient way to verify the formula's accuracy? After all, cutting out squares and trying to count how many fit inside the trapezoid to measure its area is quite a tedious task.

Yes, there is. It's called *a proof*.

Suppose your child has been given the formula for the area of a trapezoid in the following form:

$$S = \frac{h \cdot (a + b)}{2},$$

where a and b are the lengths of the lower and upper bases of the trapezoid, respectively; h is its height.

Have them show how they derived this formula. If every step is clear and obvious, then the formula is also obvious.

Proof is a reduction to the obvious, to an axiom.

An *axiom* is a statement so obvious that it requires no proof.

Although often, even obviousness fails us.

In our case, it is obvious that the area of the trapezoid is equal to the sum of the areas of two triangles and one rectangle, which are obtained after drawing two heights.

For greater clarity, we can draw a diagram.

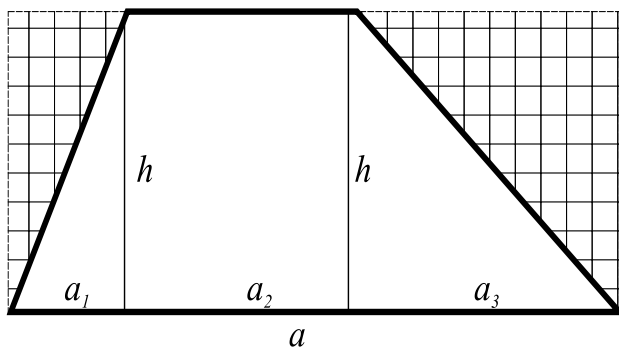


Рис. 37

So, the area of the first triangle is equal to half the area of the rectangle with sides a_1 and h .

Therefore, the area of the first triangle will be

$$S_{\Delta 1} = \frac{a_1 \cdot h}{2}.$$

The area of the rectangle in the center is obviously equal to

$$S_2 = a_2 h.$$

The area of the second triangle, accordingly, is equal to half the area of the rectangle with sides a_3 and h , or

$$S_{\Delta 3} = \frac{a_3 \cdot h}{2}.$$

Finally, the area of the trapezoid is equal to the sum of the areas obtained.

$$S = S_{\Delta 1} + S_2 + S_{\Delta 2}$$

Let's substitute the values of the areas:

$$S = \frac{a_1 \cdot h}{2} + a_2 h + \frac{a_3 \cdot h}{2} .$$

How do we add the areas of the triangles and the rectangle?

Let's reduce everything to rectangles. We have two halves of the areas of two different rectangles and one whole area () of a third rectangle. One whole consists of its two halves, i.e.

$$a_2 h = 2 \frac{a_2 \cdot h}{2}$$

Therefore,

$$S = \frac{a_1 \cdot h}{2} + 2 \frac{a_2 \cdot h}{2} + \frac{a_3 \cdot h}{2} ;$$

$$S = \frac{a_1 \cdot h + 2a_2 \cdot h + a_3 \cdot h}{2} ;$$

Let's factor out the common factor h .

$$S = \frac{h(a_1 + 2a_2 + a_3)}{2}$$

It is clear that $a_1 + a_2 + a_3 = a$.

$$S = \frac{h(a + a_2)}{2}$$

Let's replace a_2 with the parameter b . After all, they are equal as opposite sides of a rectangle. We get

$$S = \frac{h \cdot (a + b)}{2}$$

Q.E.D.

So, if the formula is obtained, it is desirable to prove it.

If the formula is unknown, then it must be derived.

Deduction and proof are two opposite processes.

Why is all this necessary?

We have a clear picture, measurable quantities, and a simple logical sequence. Years of study in a regular school with decent teachers and the solution of thousands of problems, combined with a clear understanding of the theory, gradually build the experience needed to solve more complex problems—the ones we encounter in everyday life.

35. How do you calculate the area of a regular polygon?

Let's derive a formula for calculating the area of a regular polygon. In a regular polygon, all sides and all angles are equal.

At first, it seems that the problem cannot be solved. But let's connect the vertices of the polygon to its center with line segments. We will obtain a set of congruent triangles. (Рис. 38).

Are they equal?

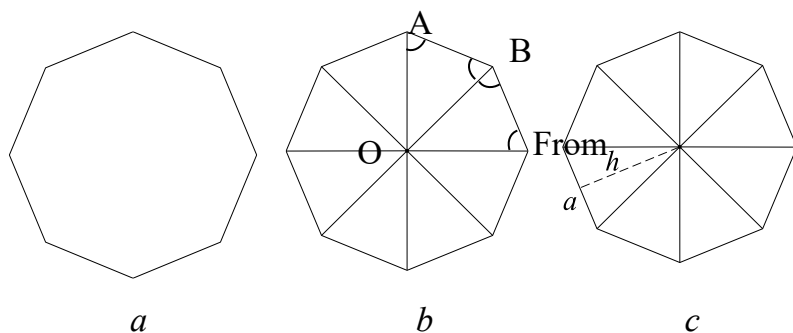


Рис. 38

Everything must be proven. Therefore, let's do it differently. Let's draw angle bisectors from the vertices—rays that bisect the angles. All the bisectors will intersect at a single point. Why?

Consider triangles AOB and BOC. They are congruent by the second criterion of triangle congruence: they have equal bases and equal base angles (Fig. 36c). Consequently, all resulting triangles will also be congruent, and

the point of intersection of the angle bisectors will be the center of a regular polygon.

To calculate the area of a single triangle, it is sufficient to measure its base a and height h . This is quite simple. The formula for the area of a triangle is well-known:

$$S_{\Delta} = \frac{a \cdot h}{2}$$

By multiplying the area of a single triangle by the number of triangles in the regular polygon, we obtain the area of the polygon. The number of triangles is equal to the number of sides of the polygon. Let the number of sides be n .

Therefore,

$$S = S_{\Delta} \cdot n \quad \text{or} \quad S = \frac{a \cdot h \cdot n}{2}$$

Once again, we have replaced *the* rather tedious *measurement* of the area of a regular polygon with a simple *calculation*. We will only add that the perpendicular dropped from the center of a regular polygon to its side is called *the apothem*.

By multiplying the length of one side (a) by the number of sides (n), we obtain the sum of the lengths of all the sides of the polygon, i.e., its *perimeter* ($P = a \cdot n$).

Consequently, the formula for the area of a regular polygon can be rewritten as:

$$S = \frac{h \cdot P}{2} .$$

The area of a regular polygon is equal to half the product of its apothem and perimeter.

36. How do you calculate the area of a circle?

A *circle* is a plane bounded by a circle .

There isn't a single familiar line segment here. What exactly do we need to measure, and which formula should we use?

A fundamentally different approach is needed here. And we are ready for it.

Let's inscribe a regular polygon in the circle . We know how to calculate the area of a regular polygon.

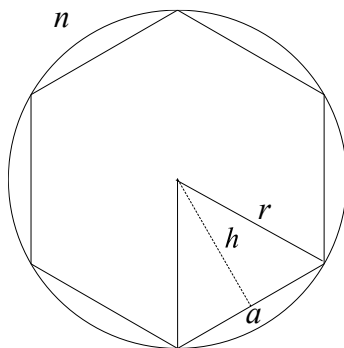


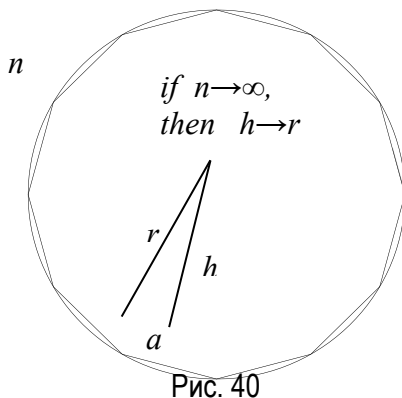
Рис. 39

It is equal to: $S = \frac{a \cdot h \cdot n}{2}$ or $S = \frac{h \cdot P}{2}$.

The diagram shows that the apothem of a regular polygon is less than the radius, the perimeter of the polygon is less than the circumference, and its area is less than the area of the circle.

What if we double the number of sides of the polygon so that it remains inscribed?

We obtain a new inscribed regular polygon, whose area is greater than that of the previous polygon, but still less than the area of a circle.



And what if we increase the number of sides to infinity?

In this case, the segments, the sides of the inscribed polygon, will shrink so much that they will turn into points and lie on the circle. Consequently, the perimeter of the polygon will be practically equal to the circumference, and its area will become equal to the area of the circle. Mathematicians say that the perimeter *tends* toward the circumference and will equal it only in the limit. But for the purposes of our study, this is not essential. It remains to add that, accordingly, the apothem will become equal to the radius.

In the formula for the area of a polygon with an infinite number of sides (), let's replace the perimeter with the circumference and the apothem with the radius:

$$S = \frac{h \cdot P}{2} = \frac{r \cdot c}{2}$$

Here, we can measure the length of the radius, but we cannot measure the circumference. Let's substitute the relationship between the circumference and the radius from the formula $c = 2\pi r$ in place of the circumference.

We obtain the formula for the area of a circle:

$$S = \frac{r \cdot \cancel{2} \pi \cdot r}{\cancel{2}} = \pi r^2$$

Amazing!

To calculate the area of a circle, it is enough to measure its radius.

The area of a circle is equal to the product of the square of the radius and the number pi.

37. How can we prove the Pythagorean theorem?

Everyone has probably heard of the Pythagorean theorem. But not everyone can state it. Even fewer can prove it. Your child will feel much calmer and more confident if they can do this. “Yes,” they’ll say, “there’s a lot I don’t know and can’t do, but I can prove Pythagoras’ theorem.”

And what does Pythagoras’s theorem state? ?

That *the square of the hypotenuse is equal to the sum of the squares of the legs*.

How can this be explained in simple terms, and how can it be proven? And what is the square of a side’s length?

If you multiply two equal factors, the resulting product is called the square of that factor. For example, $4 \cdot 4 = 16$. Therefore, 16 is the square of 4. This is written as $4^2 = 16$. It is read as: four squared equals sixteen. The small number 2 means that 4 must be multiplied by 4, i.e., two *equal factors* must be taken. But what does “square” have to do with this?

In fact, why is this operation—multiplying two equal factors—called a square? In reality, it is called *raising to a power*. We’ll discuss this in more detail in another chapter. For now, let’s say that the product of equal factors is called the power. The number of equal factors is called *the exponent*. And the factor itself is *called the base*.

Therefore, the expression $3 \times 3 \times 3 \times 3 = 81$, we can write as:

$$3^4 = 81$$

Where 3 is the base of the power, 4 is the exponent, and 81 is the power.

We know that a square is a rectangle in which all sides are equal. The area of a square is equal to the product of the lengths of its sides, which form a right angle. Since the lengths of the sides of a square are equal, we obtain the product of two equal factors.

That is, the product of two equal factors—the square of the length of a side of the square—gives the area of the square. Therefore, we say: *the area of a square is equal to the square of the length of its side!*

And what are the legs and the hypotenuse?

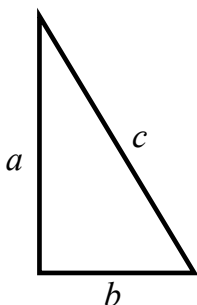


Рис. 41

In a right triangle, the sides (a and b) that form a right angle are called *the legs*. And the side (c) opposite the right angle is called *the hypotenuse*.

So, Pythagoras's theorem states: the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.

Let's prove this.

Let's extend the right triangle into a square. To do this, we'll add the length of the second leg to the length of

the first leg, and the length of the first leg to the length of the second.

We construct a square on the resulting equal sides. Let's call it the large square. The length of each of its sides is equal to the sum $(a+b)$. Рис. 42 .

Mark segments on the sides of the large square whose lengths are equal to the legs b and a . Now connect the joints of the legs with segments. These segments are the hypotenuses of congruent right triangles. They are equal to each other. The quadrilateral formed by them is a square. Let's call it the small square.

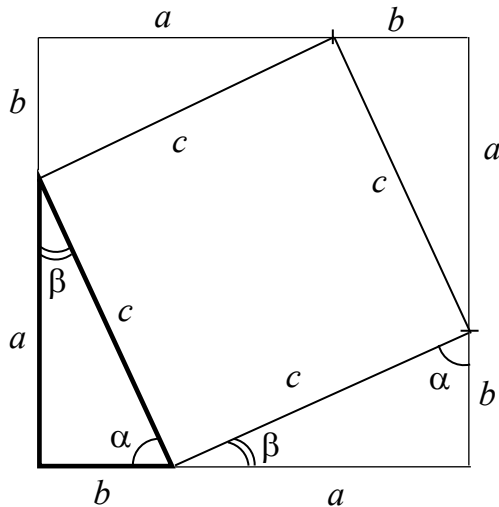


Рис. 42

Our student will nod in agreement. It will seem to him that everything is clear.

– And why is the quadrilateral formed by the hypotenuses a square?

– ?

– Are the resulting right triangles equal to each other?

– Yes!

– Why?

– By the first criterion for the congruence of triangles. Their legs are equal, i.e., the lengths of two sides and the angles between them are equal.

– Therefore, the sides of the quadrilateral formed by the hypotenuses are also equal.

Now we need to prove that all the angles in this small quadrilateral are right angles. How can we do this?

Consider the two lower triangles. The sum of the acute angles in each of them is 90° (). That is, $\alpha + \beta = 90^\circ$.

Here, the angle between the side b of the left triangle and the side a of the right triangle is a supplementary angle, i.e., it equals 180° . Therefore, the angle between their hypotenuses will be:

$$180^\circ - (\alpha + \beta) = 180^\circ - 90^\circ = 90^\circ$$

Similarly, for all pairs of hypotenuses. Therefore, the angles between them are right angles, and they form a square.

The resulting diagram shows that the area of the large square is equal to the sum of the areas of the small square and the areas of the four equal triangles. Let's write this as a formula:

$$S = S_1 + 4S_{\Delta}.$$

However, the area of the large square is equal to the square of the length of its side, and its side is equal to $(a+b)$. Therefore,

$$S = (a+b)^2.$$

The area of the small square, accordingly, is equal to the square of the length of the hypotenuse, i.e., $S_1 = c^2$.

And the area of one triangle is equal to

$$S_{\Delta} = \frac{a \cdot b}{2}$$

Therefore,

$$(a+b)^2 = c^2 + 2 \cdot \frac{a \cdot b}{2}.$$

On the left-hand side

$$(a+b)^2 = a^2 + 2ab + b^2.$$

Therefore,

$$a^2 + 2ab + b^2 = c^2 + 2ab$$

or

$$a^2 + b^2 = c^2.$$

Q.E.D.!

38. How do you apply the Pythagorean theorem?

– Why study?

– So the teacher doesn't give you a failing grade.

Therefore, if the teacher doesn't give Fs, there's no point in studying! Therefore... we can look for other ways to keep the teacher from giving Fs.

And ways are found. From a box of chocolates for her birthday to a simple bribe. And that's because the first question was answered incorrectly.

Let's try giving a different answer.

– Why study?

– To become successful and rich!

But the biographies of the successful and wealthy mostly show that they didn't really care much for school lessons. And even now, to achieve success and gain wealth, Pythagoras's theorem seems unnecessary. So why study it?

– To become a human being!

We'll leave the philosophical aspect of the question "What is a human being?" for later. Children aren't interested in that. Children aren't interested in future success or future wealth. They aren't interested in the adult future at all. They're interested in the present of childhood. Children love to play. That's why it's better to turn lessons into a game where they have to win, or into a puzzle they have to solve.

So, here's the puzzle: how do you determine the distance between two trees if there's a small marsh between them and you can't walk across it?

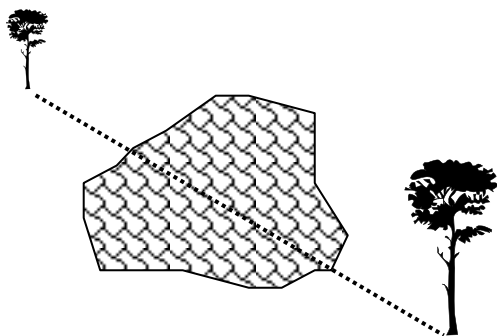


Рис. 43

A child might suggest, for example, tying a stone to a rope and throwing it across the marsh... But according to the problem's conditions, you can't cross the marsh, meaning it's impossible *to measure* the distance between the trees. And we know that if you can't measure it, you can try *to calculate it*. Calculate it using other measurements and a formula. Let them try.

Remind them of the Pythagorean theorem. Yes, they remember the theorem. But how do they apply it now?

If the child figures out how to construct a right triangle (рис. 44) and, after measuring the legs AC and BC, calculates the required distance AB, that will be a tremendous success. Such a clever child definitely needs to be praised and given a prize for winning. It doesn't matter what the prize is—let it be just a piece of candy... What matters is that the success is recognized.

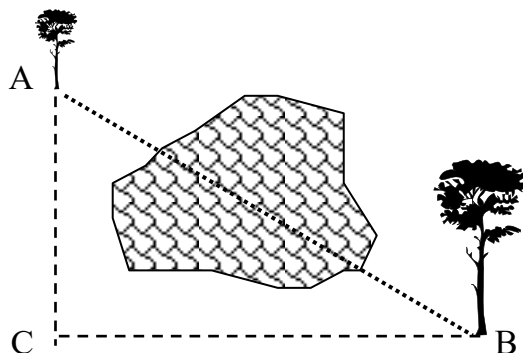


Рис. 44

After celebrating the victory, the winner should be challenged with a new problem.

It's easy to draw a right-angled triangle on paper. But how do you do it in the real world? For example, how do you build a large right-angled triangle in the yard—that is, how do you construct a right angle—if you have no protractor, no level, and no other tools... Just a long rope. Let the child think about it.

But don't keep them guessing for too long. They're unlikely to figure it out on their own. Tell them, for example, that ancient builders faced the same problem, and they solved it. They solved it using what is known as *the Egyptian triangle*. And what kind of triangle is called an Egyptian triangle?

Here we return once again to playing with numbers, or to the magic of numbers. A triangle with sides of 3, 4, and 5 units is a right triangle. This is stated by Pythagoras's theorem : the sum of the squares of the legs is equal to the square of the hypotenuse, i.e.

$$3^2 + 4^2 = 5^2$$

Therefore, the triangle is a right triangle.

Moreover, this is the only sequence of numbers differing by one from each other for which this equality holds. A triangle with sides of 3, 4, and 5 units is called an Egyptian triangle.

The sum of the sides of an Egyptian triangle () is 12 units. And we're already familiar with the number 12.

Let's take a long rope and tie knots in it so that we get 12 equal segments.



Рис. 45

Let's join the ends together to form a ring. It has the same number of divisions as a clock face—12.

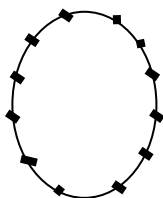


Рис. 46

Now, simply stretch the ring between pegs located at the third, fourth, and fifth knots, respectively, and we'll get a right-angled triangle.

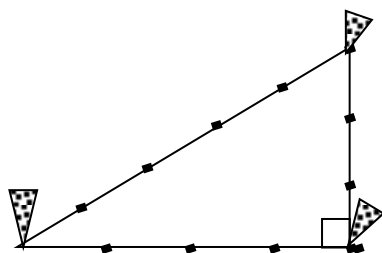


Рис. 47

Let's apply this to our problem. According to the problem's conditions, we cannot cross the swamp. But we can use all the surrounding space. We can construct a right angle elsewhere so that the legs AC and BC are extensions of the legs EC and DC .

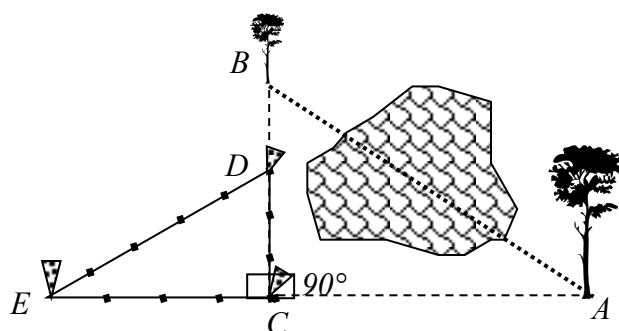


Рис. 48

39. What is a square root?

Pythagoras' theorem relates the lengths of the sides of a right triangle... Or rather, the squares of the lengths of its sides.

$$c^2 = a^2 + b^2$$

Consequently, we can obtain the equation , where the parameter will be inside the square. This is a quadratic equation.

How do we solve quadratic equations?

The method is the same. We need to separate the known and unknown quantities to different sides of the equals sign.

Let's first consider a simple case. Let's solve the equation :

$$2x^2 - 32 = 0$$

We solve it using our method: we eliminate the known term from the left side, and so on.

$$\begin{array}{rcl} 2x^2 - 32 & = & 0 \\ \underline{+32} & & \underline{+32} \\ \underline{2x^2} & = & \underline{32} \\ 2 & & 2 \\ x^2 & = & 16. \end{array}$$

Now we need to eliminate the exponent on the left-hand side. How? We need to perform the inverse operation of squaring. That is, we need to find a number x such that x^2 equals 16. This operation is called taking the square root.

Therefore, we need to take the square root of from x^2 , i.e., find a number whose square is equal to x^2 . This operation is denoted by the corresponding symbol ($\sqrt{\quad}$).

Let's take the square root of from the left and right sides of the equation:

$$\sqrt{x^2} = \sqrt{16}$$

$$x = 4$$

It seems the equation is solved. We know that the square root of 16 is 4. And $\sqrt{x^2}$ is the square root of x^2 equal to x .

Let's consider the equation $\sqrt{x^2} = x$

Let $x = -1$. We substitute this value into our equation. We get $\sqrt{(-1)^2} = -1$. Or $\sqrt{1} = -1$, i.e., $1 = -1$. This cannot be. The square of a number is always positive!

To write down the correct solution, we use a new symbol—the absolute value of a number. It is denoted by square brackets. The absolute value (modulus) of a number is always positive.

$$\sqrt{x^2} = |x|.$$

The absolute value of a number is defined as follows.

If $x > 0$, then $|x| = x$. And if $x < 0$, then $|x| = -x$. That is, if the value of x is negative, then we must place another “minus” sign in front of it to get a “plus”

For our last equation, $\sqrt{x^2} = \sqrt{16}$:

$$|x| = 4 \Rightarrow \pm x = 4 \Rightarrow x_1 = 4, x_2 = -4.$$

Our quadratic equation has two solutions, two roots.

40. How do we solve quadratic equations?

Suppose we need to solve the equation:

$$4x^2 - 12x = 0.$$

Here, we simply factor out the common factor ($4x$)

$$4x(x - 3) = 0$$

We know that the product of two expressions is zero only if one of the factors is zero.

In the first case: $4x = 0$ when $x = 0$.

In the second case: $x - 3 = 0$ when $x = 3$.

Here, the equation also has two roots: $x_1 = 0$, $x_2 = 3$.

Consider the third case.

Let

$$x^2 + 8x - 9 = 0$$

Let's do everything we can.

$$x^2 + 8x - 9 = 0$$

$$\begin{array}{r} +9 \quad +9 \\ \hline x^2 + 8x = 9 \end{array}$$

Here, factoring out x won't help. What should we do?

Note that the first term is squared, and after it comes the first term again as a factor ... This resembles the formula for contracted multiplication.

In fact, the second term on the left side of the equation can be represented as a doubled product:

$$x^2 + 8x = 9$$

$$x^2 + 2 \times 4 = 9$$

We have the square of the first number, plus twice the product of the first and second numbers. Next should

be the square of the second number. Let's add it, and, to maintain the equality, let's add it to the right side as well:

$$x^2 + 2x4 = 9$$

$$x^2 + 2x4 + 4^2 = 9 + 4^2$$

Now let's combine the first three terms on the left side of the equation into the square of the difference:

$$(x + 4)^2 = 25$$

We'll get rid of the square on the left side by taking the square root of . We'll have to do the same on the right side.

$$\sqrt{(x + 4)^2} = \sqrt{25}$$

$$|x + 4| = 5$$

$$\pm(x + 4) = 5$$

$$x + 4 = \pm 5$$

$$x = \pm 5 - 4.$$

We have two roots of the equation:

$$x_1 = 5 - 4 = 1$$

$$x_2 = -5 - 4 = -9.$$

The equation is solved.

Or it can be solved another way:

$$(x + 4)^2 = 25, \text{ so } (x + 4)^2 - 5^2 = 0.$$

Let's factor the difference of squares.

$$(x + 4)^2 - 5^2 = (x + 4 - 5)(x + 4 + 5) = (x - 1)(x +$$

9)

Therefore,

$$(x-1)(x+9)=0$$

$$x_1 = 1 \text{ or } x_2 = -9.$$

41. What is the discriminant?

In the previous lesson, we solved the quadratic equation. But isolating the square of the difference or sum every time is quite tedious and time-consuming. Is there a way to do this faster? It turns out there is.

Let's solve the quadratic equation in general form

$$ax^2 + bx + c = 0$$

We'll eliminate the factor a by dividing both sides of the equation by it and moving the free term to the right side. We get

$$x^2 + \frac{bx}{a} = -\frac{c}{a}$$

We have: the square of the first number, plus the first number x , multiplied by b/a . Let's multiply the numerator and denominator of the second term by two:

$$x^2 + \frac{2 \cdot bx}{2 \cdot a} = -\frac{c}{a}$$

We have obtained the square of the first number plus twice the product of the first (x) and the second $\left(\frac{b}{2a}\right)$ numbers. Let's add the square of the second number to both sides of the equation:

$$x^2 + \frac{2bx}{2a} + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

The first three terms can be combined into the square of the sum of two numbers:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Let's eliminate the square term on the left-hand side by taking the square root of ` ` from both sides of the equation:

$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\pm\left(x + \frac{b}{2a}\right) = \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We have obtained the formula for the roots of a quadratic equation.

Now we can quickly solve any quadratic equation . For example,

$$2x^2 + 3x + 1 = 0$$

Here, $a = 2$, $b = 3$, and $c = 1$

Therefore,

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} =$$
$$= \frac{-3 \pm \sqrt{3^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2} = \frac{-3 \pm 1}{4}$$

$$x_1 = \frac{-3 + 1}{4} = -\frac{1}{2}$$

$$x_2 = \frac{-3 - 1}{4} = -1$$

The equation is solved.

And what is the “discriminant”?

Children immediately recall this term as soon as someone starts talking to them about solving quadratic equations. Many even know how to calculate it. But when asked what it is, practically no one can answer.

Let's look again at the quadratic formula.

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here, on the right-hand side, the numerator under the square root sign contains the difference between two expressions. Consequently, for certain values *of* a , b , and c , the number under the square root may be negative. But there cannot be a negative number under the square root. In this case, we say that the equation has no roots

The difference $b^2 - 4ac$ is called *the discriminant* (from the Latin *discrimino*, “to distinguish”), i.e., it is a “distinguishing factor.” The discriminant allows us to determine whether the equation has roots, and if so, how many. Therefore, before solving the equation, we calculate the value of the discriminant.

If the discriminant is positive, then the solution continues. The equation has two roots.

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If it is negative, then we immediately conclude that there are no roots.

And if the discriminant is zero, then the equation has only one root :

$$x = -\frac{b}{2a}$$

42. What is an inequality?

– And which letters are assigned to the values of the radius, diameter, and circumference?

– These are the letters r , d , and c .

– Which letter is assigned to the length of the chord ()?

– There is no letter assigned to the chord.

– What are the letters that denote values called?

– They are called parameters.

– Does that mean the chord has no parameter?

– Well, the chord does have a length, i.e., the value , but it doesn't have an assigned letter.

– Why?

– In the same circle, the lengths of the radii, diameters, and the circumference are constant quantities. But the lengths of chords in the same circle can be different, so the letters must be different.

– Very good! Suggest a letter to denote the length of this chord.

– Let's use the letter l .

– Good. Now here's the problem: calculate the length of the diameter, the circumference, and the length of a chord in a circle with a radius of equal to 10 m .

– The diameter is 20 m . The circumference is 31.4 m , and the length of the chord... It can be any length.

– Any length? Could it be 40 m ?

– No, it can't.

– But you said the length of the chord could be any value.

– Any length, but not greater than the diameter.

- Can it be shorter?
- Yes.
- Even negative?
- No, there's no such thing as a negative length.
- How can we list all possible lengths of chords that can exist in this circle?
- That's impossible.
- What about parameters? What is the relationship between the parameters called?
- The formula !
- We have formulas relating the parameters r , d , and c . Write them down.

– There are three: $d=2r$; $c=\pi d$, and $c=2\pi r$.

– And is the parameter l (the length of the chord) related to the parameters r , d , and c ?

– No.

– But you said yourself that the length of the chord cannot be greater than the length of the diameter. So, they must be related in some way.

– Yes. But the length of the chord cannot be calculated from the length of the diameter. We only know that it is less than the length of the diameter.

– Therefore, the value of the parameter l lies within some interval. And how do we write down all possible values of the chord length for a circle of a given radius?

Intervals of parameter values are written using the familiar symbols: $>$ (greater than), $<$ (less than), $\geq$ (greater than or equal to), $\leq$ (less than or equal to), $\neq$ (not equal to). Therefore, in our case:

$$l \neq c; \quad l \leq d; \quad l \leq 2r; \quad c \geq \pi \cdot l$$

These expressions are called inequalities.

The inequality is not simply the negation of equality, but also a description of the interval of values for which this inequality holds.

Any equation becomes the inequality if we require that the variable take values that do not satisfy the given equation.

Consider the linear equation

$$2x - 12 = 0$$

The value of the variable x that satisfies this equation is 6.

And which values do not satisfy it?

All others except 6.

How can we write this down?

Clearly, for all values *of* the variable x other than 6, the expression $(2x-12)$ will be either less than zero or greater than zero.

Let's consider the first case.

$$2x - 12 > 0$$

We obtain the inequality .

The rules for solving inequalities are the same as those for solving equations. Let's solve the following inequality: .

$$\begin{array}{r} 2x - 12 > 0 \\ \underline{+12} \quad \underline{+12} \\ \underline{2x} > \underline{12} \\ 2 \qquad 2 \\ x > 6 \end{array}$$

Therefore, for all values of the variable x greater than 6, our inequality holds. Let's plot all these values on the number line.

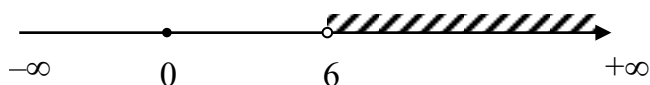


Рис. 49

The shaded area shows all solutions to our inequality. The solution to the inequality is written as follows:

$$x \in (6; +\infty)$$

This reads as follows: the variable x belongs to the interval of values from 6 to positive infinity, excluding the number 6 itself. The exclusion of the number 6 is indicated here by a parenthesis before the number 6, and on the diagram by an empty circle at the point 6.

Let's solve the second inequality: .

$$2x - 12 < 0$$

Here, $x < 6$.

Let's plot all these values on the number line.

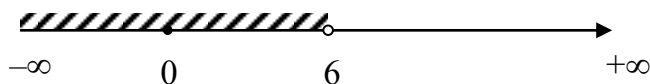


Рис. 50

The shaded area shows all solutions to the second inequality. The solution is written as:

$$x \in (-\infty ; 6)$$

This is read as follows: the variable x belongs to the interval of values from negative infinity to 6, excluding the numbers 6. The exclusion of the number 6 is indicated here by a parenthesis following the number 6, and on the graph by an empty circle at the point 6.

The union of both solutions is the set of all values of the variable x that are not solutions to the equation $2x - 12 = 0$. This union is written as

$$x \in (-\infty ; 6) \cup (6; \infty).$$

So, inequalities are solved in the same way as equations. But with one exception: when dividing or multiplying both sides of the inequality by a negative number, the inequality sign changes to the opposite.

For example, let's solve the inequality.

$$7 - \frac{x}{2} \geq 0;$$

$$-\frac{x}{2} \geq -7;$$

$$x \leq 14.$$

In the last step, we multiplied both sides of the inequality by minus two. Multiplying both sides of the inequality by a negative number causes the inequality sign to change to the opposite. It was greater than or equal to ($\geq$), and became less than or equal to ($\leq$).

On the number line, the solution to the inequality looks like this:

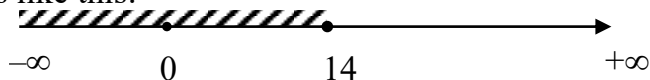


Рис. 51

The solution to the inequality is written as follows:

$$x \in (-\infty ; 14].$$

This reads: the variable x belongs to the interval of values from negative infinity to 14, including the numbers 14.

The inclusion of the number 14 is indicated here by a square bracket after the number 14, and in the diagram by a filled circle at the point 14.

We will look at solving more complex inequalities in one of the following lessons.

43. How do you calculate the probability of an event?

Game. Everyone loves to play. The main thing about a game is the uncertainty of the outcome. No one knows how it will end. If the outcome is known in advance, such a game is of no interest to anyone, and it is not a game.

A quantitative estimate of a possible outcome is called probability.

I flip a coin. Heads or tails? The result is unknown in advance. It's one of two: either heads or tails. The probability of getting "heads" is one-half, or 0.5.

I roll a die. It can land on a number from 1 to 6. The probability of rolling, for example, a 5 is one in six or one-sixth.

Let's denote the probability of event A by $P(A)$.

Let us denote the number of all possible outcomes of the event by the parameter n .

Let us denote the number of favorable outcomes by the parameter m .

Then, $P(A) = m / n$.

If $P(A) = 1$, then the event is called *certain*.

If $P(A) = 0$, then the event is called *impossible*.

If $0 < P(A) < 1$, then the event is called *a random event*.

Let's consider another example.

Suppose there are 15 balls in a box: 4 red, 5 white, and 6 blue. One ball is drawn.

Let $P(R)$ be the probability that the ball is red, $P(W)$ that it is white, and $P(B)$ that it is blue.

What is the probability of each event?

$$P(R) = 4/15$$

$$P(W) = 5/15 = 1/3$$

$$P(B) = 6/15 = 2/5.$$

The sum of all probabilities () is always equal to one.

$$P(R) + P(B) + P(C) = 4/15 + 1/3 + 2/5 = 1$$

What is the probability that the ball will be colored?

Here, the answer can be obtained in two ways.

We know that the probability that the ball will be white is $P(W) = 1/3$. Therefore, the probability that the ball will be colored is: $1 - P(W) = 1 - 1/3 = 2/3$.

Or. There are a total of $4+6=10$ colored balls. Therefore, the probability that the ball will be colored is $P(C) = 10/15 = 2/3$.

And what is the probability that the ball will be yellow?

There is no yellow ball in the urn. This event is impossible: $P(Y) = 0$.

And what is the probability that the object drawn will be a ball?

There are only balls in the box. Therefore, if we pull something out of the box, it will be a ball—this is a certain event. Its probability is equal to one: $P=1$.

44. What is volume?

Has your child watched a 3D movie?

Of course they have.

And what is 3D?

3D is short for “*three-dimensional*,” meaning three-dimensional or volumetric.

And what is volume?

A point has no size (dimension) and no definition.

A set of points located next to each other is *a straight line*... A straight line is one-dimensional. It can be conventionally described only by a point.

A part of a line bounded by two points is *a line segment*. A line segment can be measured.

A set of parallel lines is *a plane*.

The plane is two-dimensional. It can be described by a point and a line.

The part of a plane bounded by a closed curve is *the area*.

The area for a plane is the same as a segment for a line.

To measure means to compare with a standard.

Consequently, measuring an area means comparing it to a standard of area.

And what happens if we arrange a set of planes one above the other, parallel to each other? We get *a space*. Space is three-dimensional. It can be conventionally described by a point, a line, and a plane.

A space bounded by surfaces is called a *volume*.

Have the child give an example of space and volume.

Of course, the child will only think of space as cosmic (unbounded by anything), and a volume, for example, as the space of a room, bounded by the areas of the walls, floor, and ceiling.

Note that the volume of a typical room is a space bounded by rectangles. Such a volume is called *a rectangular parallelepiped*.

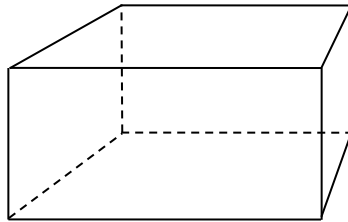


Рис. 52

Practice together to describe what is shown atрис. 52

The shortest description, i.e., *the definition*, is as follows: *a rectangular parallelepiped* is a space bounded by rectangles. In fact, if you need to bound a space with rectangles, you will need exactly six of them, and the result will always be *a rectangular parallelepiped*.

Therefore, *a parallelepiped* is a space bounded by parallelograms.

A side of a parallelepiped is called *a face*.

In a rectangular prism, all faces are parallelograms.

The sides of the faces are called the edges of the rectangular prism.

B'

C'

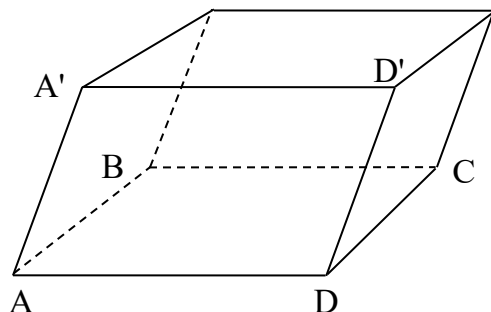


Рис. 53

In the text, a rectangular prism will be denoted by capital letters corresponding to its vertices. For example, атрис. 53 shows the rectangular prism ABCDA'B'C'D'.

How do you describe the volume?

We need to measure it. To measure means to compare it with a standard.

And what do we have as a standard for volume?

The standard for area was a portion of the plane bounded by a square with a side length of 1 *m*, i.e., one square meter (1 m^2).

The standard for volume will be a cube with an edge length of 1 *m*, i.e., one cubic meter (1 m^3). A cube has six faces, and each face is a square.

A *cube* is a rectangular prism with square faces.

Ask your child to describe the volume of a room. Ask what needs to be done to measure the volume of a room.

They will, of course, start saying that you need to take a ruler and first measure the length and width, then measure the height, and multiply everything together. That's how they were taught in school. But remind them

that the task was to measure the volume, not to calculate it.

If they can't figure it out, you'll say in a dramatic tone that to do this, they'll have to move all the furniture out, then build a cube out of plywood with a length, width, and height of , each 1 m .

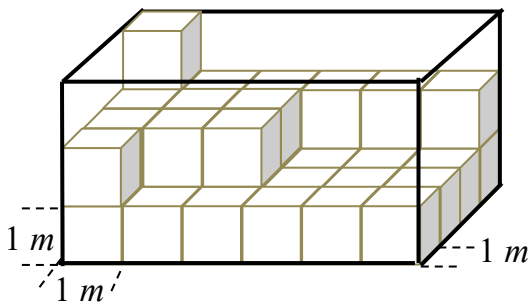


Рис. 54

And then they'll have to figure out how many of these cubes it would take to completely fill the entire room.

He'll be impressed.

Is there a way to simplify this?

Yes, we can. Let's conduct a thought experiment.

Ask how many cubes it would take to fill the entire floor. We know how to calculate the floor area. The floor area shows how many squares with a side of one meter would be needed to completely fill the floor. But the base of a cube with an edge of 1 m also has an area of exactly 1 square meter. Therefore, the number of cubes that will cover the floor will be equal to the floor area. And this is just one layer 1 m high. The number of layers will be equal to the height of the room in meters.

Have the child measure the length and width of the room by pacing. Let's say two of his average steps are 1 *m*. Suppose the room is 5 meters long and 4 *meters* wide. Now let's calculate the floor area . He'll be able to do this easily.

$$S = a \cdot b = 4 \times 5 = 20 \text{ (m}^2\text{)}$$

How many cubic meters will be needed to cover the entire floor? That's right, 20 as well.

And how many more cubic meters will be needed to fill the entire room? The height of the cubes is 1 *m*. And what is the height of the room? Let's say its height is 3 *m*. One meter of height accounts for 20 cubic meters, and three meters account for 60.

Therefore, to calculate the volume of a rectangular prism, we need to measure the lengths of its edges that form right angles with each other . Their product gives the volume, i.e., $V = a \times d \times h$

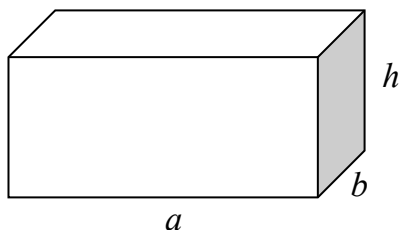


Рис. 55

V – the parameter , denoting volume (the first letter of the English word "volume")

Volume of a cube:

$$V = a \times a \times a = a^3$$

Now it is clear why the result of multiplying three equal factors is called the cube of a number.

Volume of a rectangular prism: $V = S \times h$

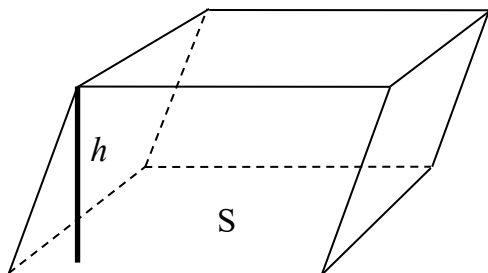


Рис. 56

Here, h is the height of the rectangular prism—the perpendicular line dropped from the vertex of the prism to its base; S is the area of the base of the rectangular prism.

So, a rectangular parallelepiped has a volume. But it also has a surface area. How do you calculate the surface area of a rectangular parallelepiped?

Let the child think about this problem on their own and derive the formula themselves; if necessary, ask them a few leading questions.

45. Formulas for the volumes of various shapes

Volume is the space bounded by surfaces.

A volume bounded by polygons is called *a polyhedron*.

A polyhedron with two parallel and equal faces is called *a prism*.

Volume of a prism: $V = Sh$.

A polyhedron in which all but one of the faces are triangles sharing a common vertex is called *a pyramid*.

Volume of a pyramid: $V = \frac{1}{3} Sh$

A pyramid with a triangular base is called *a triangular pyramid*.

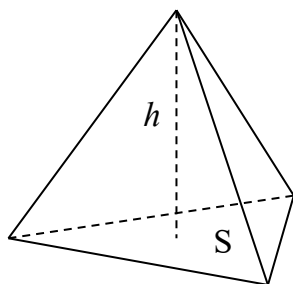
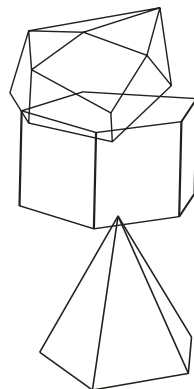


Рис. 57

That leaves the cone and the cylinder.

Let's imagine that the pyramid has a regular polygon with an infinitely large number of sides at its base. Such a

base would be no different from a circle. The pyramid turns into a cone.

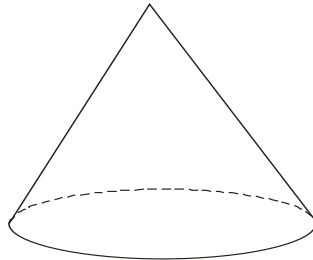


Рис. 58

Volume of a cone: $V = \frac{1}{3} Sh = \frac{1}{3} \pi r^2 h$

Now let's imagine a prism with a regular polygon as its base, having an infinitely large number of sides. Again, we get a circle as the base. The prism turns into a cylinder.

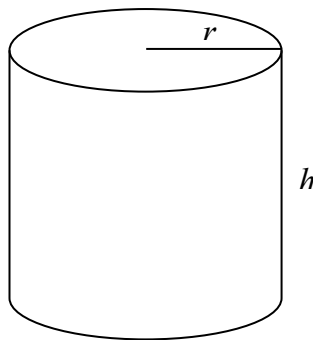


Рис. 59

Ask your child to describe what they see. (Рис. 59). If they say they can't describe it, encourage them to try describing it incorrectly at first. An incorrect description is still a description. It can be corrected later.

Have them try to recall the description of a cube or a rectangular prism.

A cylinder is a space bounded by two parallel circles and... a curved rectangle? Not very complicated.

Let's try a different approach. Recall how we described volume. Similarly, we get: a cylinder is a set of parallel circles strung on a line passing through their centers.

That's better.

The top and bottom circles are called *the bases of the cylinder*.

The surface of the cylinder, formed by the set of its circles, is called *the lateral surface*.

The perpendicular line connecting the planes of the cylinder's bases is called its *height*.

It is impossible to measure the volume of a cylinder using cubes. But it can be calculated.

There is a circle at the base of the cylinder. We know how to calculate the area of a circle: $S = \pi r^2$.

The volume of the cylinder will be equal to the product of the area of its base and its height.

$$V = S \cdot h = \pi r^2 h.$$

The surface area of the cylinder is the sum of the areas of its bases and the area of its lateral surface. The lateral surface is a right triangle with sides equal to the height and the circumference.

The area of the base of the cylinder $S_1 = \pi r^2$.

The area of the lateral surface of the cylinder

$$S_2 = c \cdot h = 2\pi r \cdot h.$$

Therefore, the total surface area of the cylinder is:

$$S = 2S_1 + S_2 = 2\pi r^2 + 2\pi r h = 2\pi r(r + h).$$

Now let's consider a ball. Ask the child to describe the surface of the ball. At first, it will seem like there's nothing to latch onto here.

Have them recall the definition of a circle. They'll do this right away. Now, explain that a more precise mathematical definition would be: *a circle is the set of points in the plane equidistant from a given point.*

Point out that, in a certain sense, the surface of a ball is also a set of points in space equidistant from a given point. This set of points is called *a sphere*.

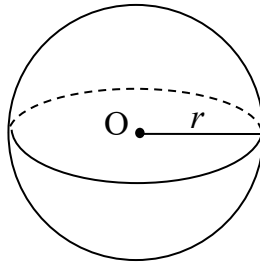


Рис. 60

So, *a sphere is the area, equidistant from a given point (or a set of points in space equidistant from a given point).*

A point equidistant from the sphere is called its center.

The line segment connecting a point on the sphere to its center is called the radius of the sphere.

Finally, *a sphere is the space bounded by a sphere.*

The surface area of a sphere is:

$$S = 4\pi r^2.$$

The volume of the sphere:

$$V = \frac{4}{3}\pi r^3$$

You can ask a series of questions that the child will have to answer on their own.

- Does the sphere have a diameter ?
- How can you measure the diameter of the sphere?
- Is it possible to measure the radius of the ball?
- How do you calculate the radius of the ball?

We have obtained a set of figures whose surface areas and volumes are calculated using the corresponding formulas. Moreover, these formulas are related to one another through common parameters. Consequently, we can construct a set of equations that allow us to calculate the required parameter using its relationships with other parameters. It is important for the child to understand that **these relationships can be constructed independently!**

46. What is the power of a number?

In the previous chapters, we referred to the addition of equal terms as multiplication (p.35), and the result of multiplication as a product .

We called the product of two equal factors the square of a number: $a \cdot a = a^2$.

We called the product of three equal factors the cube of a number: $a \cdot a \cdot a = a^3$.

And what will the product of four equal factors be called? Or of any number of equal factors?

It is called the power of a number.

The power of a number is the product of equal factors. The factor itself , here, is called the base of the power, and the number of factors is called the exponent .

For example, $3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$. Here, 3 is the base of the power; 4 is the exponent ; 81 is the power.

$$a^4 = a \cdot a \cdot a \cdot a$$

This is a raised to the fourth power. Here, a is the base, and 4 is the exponent .

Multiplying powers with the same base is straightforward:

$$3^2 \cdot 3^3 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^5, \text{ i.e., } 3^2 \cdot 3^3 = 3^{2+3} = 3^5.$$

Let's write this in terms of parameters.

$$a^n \cdot a^m = a^{n+m}$$

Let's consider division.

$$\frac{3^7}{3^5} = \frac{\cancel{3} \cdot \cancel{3} \cdot \cancel{3} \cdot \cancel{3} \cdot \cancel{3} \cdot 3 \cdot 3}{\cancel{3} \cdot \cancel{3} \cdot \cancel{3} \cdot \cancel{3} \cdot \cancel{3}} = 3^2$$

Or

$$\frac{3^7}{3^5} = 3^{7-5} = 3^2$$

Let's write this in parametric form.

$$\frac{a^n}{a^m} = a^{n-m}$$

After this introduction, ask your child to raise a power to another power. For example, have your child convert the expression $(3^2)^4$ to a power with base 3.

The result should be:

$$(3^{(2)})^{(4)} = (3^{(2)}) \cdot (3^2) \cdot (3^2) \cdot (3^2) = 3^{2+2+2+2} = 3^{2 \cdot 4} = 3^8$$

or

$$(3^2)^4 = 3^{2 \cdot 4} = 3^8$$

Then ask him to write the same expression using variables.

$$(a^n)^m = a^{n \cdot m}$$

That was all theory. Now ask them to calculate the value of the expression $(5^6)^3 : (5^2)^8$?

$$(5^6)^3 : (5^2)^8 = \frac{5^{18}}{5^{16}} = 5^2 = 25.$$

Now ask them to calculate the value of the following expression

$$(25^6)^3 : (5^7)^5$$

Let's expand the parentheses to get:

$$(25^6)^3 : (5^7)^5 = 25^{6 \cdot 3} : 5^{7 \cdot 5} = 25^{18} : 5^{35}$$

Now ask yourself, are the bases of the powers equal?
Of course not.

Can we reduce them to the same base?

Yes, we can. We need to think of 25 as five squared.

We get:

$$25^{18} : 5^{35} = (5^2)^{18} : 5^{35} = \frac{5^{2 \cdot 18}}{5^{35}} = \frac{5^{36}}{5^{35}} = 5^{36-35} = 5$$

To solidify the ability to follow the rules, you need to work through many examples. You'll find them in your school textbook. And we'll ask the already quite exhausted child: Are there negative exponents? He'll look up to the sky in despair, but don't back down. Tell him that, of course, there are, and we've just learned about them.

"Where?" they'll exclaim.

Let's look at the last rule again

$$\frac{a^n}{a^m} = a^{n-m}$$

It can be written as follows:

$$\frac{a^n}{a^m} = a^{n-m} = a^{n+(-m)} = a^n \cdot a^{-m}$$

That is, $\frac{a^n}{a^m} = a^n \cdot a^{-m}$

Divide both sides by $a^{(n)}$ and we get

$$\frac{1}{a^m} = a^{-m}, \text{ and conversely, } \frac{1}{a^{-m}} = a^m$$

Therefore, a negative exponent denotes not simply the number of equal factors, but the number of equal factors that are inverses of the given base! Therefore,

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9} \quad \text{or} \quad \frac{1}{2^{-4}} = 2^4 = 16$$

Let's add one more rule: the product of s of powers with equal exponents is equal to their product raised to that power, i.e.,

$$a^n \cdot b^n = (a \cdot b)^n.$$

Conversely, the power of a product is equal to the product of the powers:

$$(a \cdot b)^n = a^n \cdot b^n$$

Let the child prove the validity of this rule on their own.

Using the last rule, let's evaluate the expression

$$\frac{6^7 \cdot 8^7}{48^6} = \frac{(6 \cdot 8)^7}{48^6} = \frac{48^7}{48^6} = 48^{7-6} = 48$$

Now, ask the student to perform the calculation, making the transformation only in the denominator.

$$\frac{6^7 \cdot 8^7}{48^6} = \frac{6^7 \cdot 8^7}{(6 \cdot 8)^6} = \frac{6^7 \cdot 8^7}{6^6 \cdot 8^6} = 6^{7-6} \cdot 8^{7-6} = 6 \cdot 8 = 48$$

And how do you raise a fraction to a power?

It's very simple, because raising to a power is the multiplication of equal factors.

$$\left(\frac{2}{3}\right)^3 = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{2 \cdot 2 \cdot 2}{3 \cdot 3 \cdot 3} = \frac{2^3}{3^3} = \frac{8}{27}$$

Therefore,

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} = a^n \cdot b^{-n}$$

So far, our exponents have been positive and negative integers. But what is the power of a number if its exponent is zero? Let's consider, for example, three to the power of zero in the form

$$3^0 = 3^{2-2} = \frac{3^2}{3^2} = 1.$$

Therefore, *any* number raised to the power of zero is equal to one, i.e.

$$a^0 = 1.$$

You can also raise complex expressions to a power. Ask your child to expand the parentheses in the expression $(a+b)^2$.

If they write that $(a+b)^2 = a^2 + b^2$, ask them to check it by substituting specific numbers. For example,

$$(3+5)^2 = 3^2 + 5^2 = 9+25=34.$$

$$\text{But, in fact, } (3+5)^2 = 8^2 = 64.$$

The answers don't match!

Let them recall once again what it means to square an expression. The exponent "2" means that we have two equal factors.

Therefore,

$$(a+b)^2 = (a+b) \cdot (a+b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2$$

To avoid having to do these lengthy calculations every time, you can remember that *the square of the sum of two numbers is equal to the square of the first number, plus twice the product of the first and second numbers, and plus the square of the second number*. That is, you can write it down immediately as:

$$(a+b)^2 = a^2 + 2ab + b^2$$

Let's perform a calculation using this formula

$$(3+5)^2 = 3^2 + 2 \cdot 3 \cdot 5 + 5^2 = 9 + 30 + 25 = 64.$$

Now everything adds up.

Similarly,

$$(a-b)^2 = a^2 - 2ab + b^2$$

The square of the difference of two numbers is equal to the square of the first number minus twice the product of the first and second numbers, plus the square of the second number.

As homework, ask your child to independently derive the formula for the shortened multiplication of the cube of the sum and the cube of the difference.

$$\begin{aligned} (a+b)^3 &= (a+b) \cdot (a+b)^2 = (a+b) \cdot (a^2 + 2ab + b^2) = \\ &= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3 = a^3 + 3a^2b + 3ab^2 + b^3 \end{aligned}$$

47. Simplifying Fractions and the Shortcut Multiplication Formula

Simplify the expression.

Many problems in textbooks begin with this phrase. What does it mean, and why is it necessary?

Once upon a time, long ago, when all calculations were done by hand on paper, simplifying an expression beforehand was absolutely necessary to speed up the calculations. Today, in the age of computers, there is no particular need for this. But we need to instill in the child the ability to simplify a complex problem; we need to show that it is possible at all. So let's play around with some examples.

Let the child try to expand the parentheses in the product.

$$(a+b)(a-b).$$

He will be able to do this.

He should demonstrate, with surprise and some delight, that

$$(a+b)(a-b) = a^2 - \cancel{ab} + \cancel{ab} - b^2 = a^2 - b^2$$

Be amazed along with him and write down this wonderful formula separately:

$$(a+b)(a-b) = a^2 - b^2$$

We have obtained the difference of the squares of two numbers. Therefore, we can now factor the difference of squares:

$$a^2 - b^2 = (a+b)(a-b)$$

Have the child read this expression.

They will say the following: “‘A’ squared minus ‘B’ squared equals ‘A’ plus ‘B’ multiplied by ‘A’ minus ‘B’.”

Excellent!

Now have them read the same thing, but using terms instead of variables.

The child pauses to think.

Work with them to formulate the following sentence: *“The difference of the squares of two numbers is equal to the product of their sum and difference.”* They’ll be impressed.

Now ask them to simplify the expression.

$$\frac{a^2 - b^2}{(a + b)^2}.$$

The child will stare at it for a long time but won’t dare to do anything. Remind them that they should simply do what they can and not do what they can’t. Can they factor the difference of squares in the numerator and the square of the sum in the denominator? Of course they can. Is it necessary to do this? Yes, it is. Will it help? We don’t know yet, but let’s do it!

$$\frac{a^2 - b^2}{(a + b)^2} = \frac{(\overline{a+b})(a-b)}{(\overline{a+b})(a+b)} = \frac{(a-b)}{(a+b)}$$

That turned out great!

Therefore, if we need to evaluate the expression

$$\frac{75^2 - 25^2}{(75 + 25)^2}$$

then after simplifying, it wouldn’t be hard to do:

$$\frac{75^2 - 25^2}{(75 + 25)^2} = \frac{(\overline{75+25})(75-25)}{(\overline{75+25})(75+25)} = \frac{75-25}{75+25} = \frac{50}{100} = \frac{1}{2}$$

Let’s try another one.

We need to simplify the expression

$$\frac{x^2 - 2xy + y^2}{x - y}.$$

There's nothing we can do about the denominator. But the numerator is the square of the difference between two numbers. Let's expand it using the well-known formula for shortened multiplication: $(a-b)^2 = a^2 - 2ab + b^2$.

$$\frac{x^2 - 2xy + y^2}{x - y} = \frac{(x - y)^2}{x - y} = \frac{(\cancel{x-y})(x - y)}{\cancel{x-y}} = x - y.$$

Great!

Now let's try a more complicated expression.

Let him calculate the value of the expression

$$a(36a^2 - 25) \left(\frac{1}{6a + 5} - \frac{1}{6a - 5} \right), \text{ where } a = 36.7$$

If we substitute the value of a directly, the calculation will be too cumbersome. First, we need to simplify the expression.

Ask: What numbers, when squared, give 36 and 25, respectively? What happens if we replace the numbers with their squares in the first parenthesis?

$$36a^2 - 25 = 6^2 a^2 - 5^2 = (6a)^2 - 5^2$$

We end up with a difference of squares. Have him factor it:

$$(6a)^2 - 5^2 = (6a - 5)(6a + 5)$$

Now we just need to substitute the result into the main expression.

$$a(6a - 5)(6a + 5) \left(\frac{1}{6a + 5} - \frac{1}{6a - 5} \right)$$

What to do next. He will most likely be tempted to subtract the terms in parentheses separately. But it will be easier to expand the parentheses and perform the multipli-

cation. By canceling out the common factors in the numerator and denominator, we get

$$\begin{aligned}
 & a(6a-5)(6a+5) \left(\frac{1}{6a+5} - \frac{1}{6a-5} \right) = \\
 & = a(6a-5)(\cancel{6a+5}) \frac{1}{\cancel{6a+5}} - a(\cancel{6a-5})(6a+5) \frac{1}{\cancel{6a-5}} = \\
 & = \cancel{6a^2} - 5a - \cancel{6a^2} - 5a = -10a
 \end{aligned}$$

Now we can substitute the value $a = 36.7$

$$-10a = -10 \times 36.7 = -367.$$

But what if the exponent is a fraction?

We need to start a new lesson.

48. Fractional exponents

Exponentiation is a completely independent mathematical operation. This means there must be an inverse operation. The inverse of exponentiation is called taking the root of the corresponding power. Therefore, we can call the factorization of a power into equal factors “taking the root.”

For example, calculating a number whose square is 16 is called taking the square root of 16. In this case, we need to find the base of the power. That is, we need to factor 16 into two equal factors. What will this factor be?

This operation is written using a special symbol, $\sqrt{\quad}$, which is called the square root of or the square root.

$$\sqrt{16} = \sqrt{4 \cdot 4} = \sqrt{4^2} = 4$$

Have the child read this expression.

He will read: “The square root of 16 is 4.” Very good. Now, ask him to read the same expression without saying the word “root.” Help him arrive at: “Find the number, whose square is 16.” Show that the short notation $\sqrt{16}$ replaces the six-word sentence.

Accordingly, calculating the number whose cube is, for example, 27, is called taking the cube root of 27. This operation is written using the special symbol $\sqrt[3]{\quad}$, which is called the cube root or the cube root.

$$\sqrt[3]{27} = \sqrt[3]{3 \cdot 3 \cdot 3} = \sqrt[3]{3^3} = 3$$

Here is how an operation involving the calculation of a number whose fourth power is 81 is written:

$$\sqrt[4]{81} = 3$$

Here, the fourth root of from the number 81 is equal to three.

Let's write these operations using parameters.

$$\sqrt[m]{a} = b$$

The root may also involve a power of a power, i.e., $\sqrt[m]{a^n}$, for example, $\sqrt[5]{7^4}$. The value of this expression is difficult to calculate. But ask the child to calculate the value of the expression $\sqrt[8]{7^8}$. Help the child figure out on their own that the result will be seven.

Similarly, the square root of can be raised to a power: $(\sqrt[n]{a})^m$.

For example, $(\sqrt[3]{7})^2$. Here, try to help the child recognize the familiar concept of exponentiation. Whatever is under the exponent 2 must be multiplied by itself.

You can demonstrate this as follows:

$\square^2 = \square \cdot \square$; $\Delta^2 = \Delta \cdot \Delta$; $(\otimes + \otimes)^2 = (\otimes + \otimes) \cdot (\otimes + \otimes)$ and so on.

Therefore,

$$(\sqrt[3]{7})^2 = \sqrt[3]{7} \cdot \sqrt[3]{7} = \sqrt[3]{7 \cdot 7} = \sqrt[3]{49}$$

To perform operations on roots, it is convenient to represent them as powers:

$$\sqrt[m]{a^n} = a^{\frac{n}{m}}$$

For example,

$$\sqrt[3]{5^2} = 5^{\frac{2}{3}}.$$

A fractional power is always a root of. For example,

$$7^{1.5} = 7^{1\frac{1}{2}} = 7^{\frac{3}{2}} = \sqrt{7^3} = 7\sqrt{7}$$

Try to simplify the expression $\sqrt[3]{5^2} \cdot \sqrt[6]{5^5}$

Since our theoretical course was quite brief, this task will seem difficult to the child at first. But tell them that they just need to confidently follow the known rules.

Let's represent the roots as powers. And when multiplying powers with the same base, their exponents add together. Therefore,

$$\sqrt[3]{5^2} \cdot \sqrt[6]{5^5} = 5^{\frac{2}{3}} \cdot 5^{\frac{5}{6}} = 5^{\frac{2}{3} + \frac{5}{6}} = 5^{\frac{1}{2}} = \sqrt{5}$$

Ask them to evaluate the expression $\sqrt[3]{7} \cdot 49^{\frac{5}{6}}$.

The result should be:

$$\begin{aligned} \sqrt[3]{7} \cdot 49^{\frac{5}{6}} &= 7^{\frac{1}{3}} \cdot 49^{\frac{5}{6}} = 7^{\frac{1}{3}} \cdot (7^2)^{\frac{5}{6}} = 7^{\frac{1}{3}} \cdot 7^{2 \cdot \frac{5}{6}} = 7^{\frac{1}{3}} \cdot 7^{\frac{5}{3}} = \\ &= 7^{\frac{1}{3} + \frac{5}{3}} = 7^{\frac{6}{3}} = 7^2 = 49. \end{aligned}$$

Let's evaluate the expression

$$\frac{6^{\sqrt{5}} \cdot 8^{\sqrt{5}}}{48^{\sqrt{5}-1}} = \frac{(6 \cdot 8)^{\sqrt{5}}}{48^{\sqrt{5}} \cdot 48^{-1}} = \frac{\cancel{48^{\sqrt{5}}} \cdot 48^1}{\cancel{48^{\sqrt{5}}}} = 48$$

How do we take the root of from the fraction ?

Since we can represent the root of as a fractional power, and we already know how to work with powers, we get:

$$\sqrt[3]{\frac{8}{27}} = \left(\frac{8}{27}\right)^{\frac{1}{3}} = \frac{8^{\frac{1}{3}}}{27^{\frac{1}{3}}} = \frac{\sqrt[3]{8}}{\sqrt[3]{27}} = \frac{2}{3}.$$

Or simply

$$\sqrt[3]{\frac{8}{27}} = \frac{\sqrt[3]{8}}{\sqrt[3]{27}} = \frac{2}{3}$$

Let's write this in terms of variables:

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Now, let the child try to calculate the value of the expression

$$\frac{b^{3\sqrt{2}+2}}{(b^{\sqrt{2}})^3}, \text{ with } b=6$$

Let them do what they can.

In the numerator, the exponent is the sum of . So, we can factor the exponent.

And in the denominator, we have a power raised to a power, so the exponents must be multiplied together, i.e.

$$\frac{b^{3\sqrt{2}+2}}{(b^{\sqrt{2}})^3} = \frac{\cancel{b^{3\sqrt{2}}} \cdot b^2}{\cancel{b^{3\sqrt{2}}}} = b^2$$

Now we can substitute the value $b=6$.

$$b^2 = 6^2 = 36$$

Is everything clear? Maybe. Let's check. Let our student solve the equation :

$$4^{2x-5} = 64$$

Of course, he'll start to get upset: he's never seen equations like this before. No big deal. Let him rewrite the given equation into a form he's seen before!

We have a power on the left-hand side. Can we also express the right-hand side as a power?

Yes. $64 = 4^3$.

Let's substitute this value into the right-hand side.

We get

$$4^{2x-5} = 4^3$$

Now, let him compare the bases of the powers. They are equal to each other. On the left side, the base is 4, and on the right side, it is also 4. Therefore, the exponents must also be equal. Let's write this down.

$$2x-5=3$$

He can already solve this equation () on his own.

All of the above steps apply when the bases of the powers are positive. If the base is negative, raising to a fractional power requires a special approach.

49. What is a logarithm?

Let's write down the power of a number and the root of a number .

$$b = a^n ; a = \sqrt[n]{b}$$

Note that $b > 0$ and $a > 0$.

- How many parameters are there here?
- Three.
- Which ones?
- a, b, n .
- And how many formulas?
- Two.
- We're missing one formula. Which one?

Probably the one that would help calculate the exponent n . We need to calculate how many equal factors there were in this power. Therefore, we need to perform the inverse operation of raising to a power. That is, we need to factor the power into equal factors and count their number.

But how do we write the expression in parametric form: *find the exponent n to which the base a must be raised to yield the power b ?*

The symbol denoting this operation is called *a logarithm* . It is written as

$$\log_a b = n$$

It is read as: the logarithm b to the base a is equal to n .

Logarithm calculation is the decomposition of a number into equal factors. The result of calculating the logarithm is the exponent. In our case, the exponent of the

power n , to which the base a must be raised to yield the power b .

Let's add this formula to the two we had at the beginning, and we'll get three formulas relating the three parameters.

$$b = a^n ; a = \sqrt[n]{b} ; n = \log_a b.$$

Did our student understand what the logarithm of a number is?

Maybe. But it would be good to check. Let him calculate the value of the expression:

$$5^{\log_5 17}.$$

He will say that this is impossible.

– The exponent of the number five is the logarithm of . Is that clear?

– Yes.

– What is it equal to? Read the exponent of .

– It equals the logarithm of 17 to the base 5.

– And what is that?

– I don't know!

– I don't know either. But the logarithm of means the exponent .

– Yes.

– In our case, what is the exponent of ?

– The number 5.

– And what should we get if we raise the number 5 to that power?

– 17.

– Therefore, the logarithm of 17 to base 5 is equal to the exponent to which 5 must be raised to equal 17.

– Yes...

– Please repeat that.
– The logarithm of 17 to base 5 is equal to the exponent to which 5 must be raised to equal 17.

– Excellent. Now look at the whole expression again. What do you see?

– I see the number 5, and its exponent is the logarithm .

– In other words, the 5 has an exponent of , which is also an exponent. After all, the logarithm is an exponent.

– Yes.

– So, the exponent of 5 in is another exponent.

– Yes.

– Do we know what it is?

– No.

– What do we know about it?

– We know that if we raise the number to the power of 5, the result should be 17.

– Well, there you go! Do you understand?

– No...

– In general terms, the number 5 has an exponent of such that if you raise the number to the power of 5, you get the number 17. See?

– Yes.

– So, what is the value of the entire expression?

– $5^{\log_5 17} = 17$.

– Well done!

In general form, this is written as:

$$a^{\log_a b} = b$$

The formula can be considered a logarithmic formula: **the logarithm** of a number b to base a is defined as the

exponent to which the base a must be raised to obtain the power b .

So, we have the following three expressions

$$b = a^n; \quad a = \sqrt[n]{b}; \quad n = \log_a b.$$

Clearly, all three are the same formula, only each time a different parameter is calculated.

– Is everything clear?

– Yes.

– Let's check. Solve the equation

$$\log_3(x - 7) = 4$$

– I haven't seen equations like that before.

– Try to rewrite this equation, , into a form you've seen before.

– How?

– Just rewrite the same expression in a different form. For example, as a power.

$$3^4 = x - 7$$

$$81 = x - 7$$

$$x = 88$$

Great!

For logarithms, you can derive the same formulas as for powers and roots.

1. The logarithm of to the base 1 is zero:

$$\log_a 1 = 0.$$

2. If the logarithm of a number to base 1 is equal to 1, then the number itself, , is equal to the base of the logarithm:

$$\log_a a = 1.$$

3. The logarithm of a product is equal to the sum of the logarithms of the factors:

$$\log_a xy = \log_a x + \log_a y$$

Here, let our student prove this. Just point out that we need to use the fundamental logarithm formula, which allows us to express the number in terms of its logarithm. Let's express x and y in terms of their logarithms:

$$x = a^{\log_a x}; \quad y = a^{\log_a y}.$$

Let's calculate the product $x \cdot y$, substituting the values of the factors:

$$x \cdot y = a^{\log_a x} \cdot a^{\log_a y} = a^{\log_a x + \log_a y}$$

i.e.

$$xy = a^{\log_a x + \log_a y}$$

Now ask: to what power must the number a be raised here to obtain the product xy ? Of course, to the power $(\log_a x + \log_a y)$.

And what should the exponent of a be to get the product xy ? Of course, the exponent of a must be the logarithm of xy to base a , i.e.

$$a^{\log_a xy} = xy$$

Let's substitute this value *of* xy into the previous formula:

$$a^{\log_a xy} = a^{\log_a x + \log_a y}$$

If the bases of the powers are equal, then the exponents are also equal, i.e.,

$$\log_a xy = \log_a x + \log_a y$$

Q.E.D.!

4. The logarithm of the quotient is equal to the difference of the logarithms of the dividend and the divisor.

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

This is proven in a similar way to the previous case.

$$\frac{x}{y} = \frac{a^{\log_a x}}{a^{\log_a y}} = a^{\log_a x - \log_a y}$$

Here too, the exponent, to which the number a must be raised to obtain the ratio x/y , can be written as the logarithm of this ratio to base a :

$$a^{\log_a \frac{x}{y}} = \frac{x}{y}.$$

Therefore,

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

5. The logarithm of a power is equal to the product of the exponent and the logarithm of the base raised to that power.

$$\log_a x^n = n \log_a x$$

We prove it in the same way. We know that:

$$x = a^{\log_a x}$$

Let's raise both sides to the power of n .

$$x^n = (a^{\log_a x})^n;$$

$$x^n = a^{n \log_a x}$$

Therefore, by the definition of a logarithm, the exponent, to which the number a must be raised to obtain x^n , is $n \log_a x$. Let's write this down.

$$\log_a x^n = n \log_a x$$

50. Lessons and Puzzles

You have probably already realized that a certain method was used in all the previous lessons. What is this method?

In the famous Panchatantra (Sanskrit for “Five Books”)—an ancient Indian book of wisdom—there is a story that reveals the essence of this method. The book begins as follows:

“Long ago in southern India there was a city called Pataliputra (now known as Patna), ruled by the just king Sudarsana. He was revered not only by the people of his kingdom but also by the kings of neighboring states. They often came to him to pay their respects and bring rich gifts. His subjects also loved their king dearly, but despite this, the raja was sad. He worried about his sons, who refused to study and behaved very badly. For it is said that youth, wealth, power, and recklessness, as well as indiscriminateness, can ruin a person devoid of virtue. Just as sweet river water becomes tasteless when it reaches the sea, so too do all these things become vices when possessed by an unkind person. And it is also said: “Just as there is no light from a multitude of stars, so too there is no benefit from foolish sons. But like the moon illuminating the night sky, a single worthy son brings glory to the entire family.”

Grief increasingly gnawed at the king’s heart. Finally, unable to bear his children’s foolishness any longer, he summoned a secret council. “O wisest ones,” he addressed his ministers, “you know that in the material world, happiness comes only from wealth, good health, a

beautiful and eloquent wife, an obedient son, and knowledge. A wise man must strive for knowledge as if he were to live forever. A son who is neither a saint nor a scholar is a disgrace to the whole family. The thought of my children gives me no peace. By refusing to study, they grow more foolish day by day. What good are foolish and disobedient sons! A son who has not been born or who is dead is better than a foolish one, for the first two cause suffering only once, while a fool causes it at every turn. Just as a crow is not honored among swans, so too does an uneducated son find no respect in the company of learned men. Parents who do not give their son an education are his enemies. I must do something, and so I have decided to turn to you for help. Does any of you know a guru who could teach my children the nobility and wisdom that worthy sons of a raja ought to possess? Just as a shard of glass appears to be a treasure when placed next to gold, so too does a fool gain wisdom through association with learned people. A person degenerates when associating with lowly people, remains unchanged in the company of equals, and improves by associating with those who surpass him in some way. I will spare no expense as a reward for his labor.”

But the advisors and ministers remained silent. Finally, one of those present stood up and said, “The holy man Vishnusharma lives in the city. He is very wise. I think he will surely be able to teach the princes the art of righteous living. Invite him to the palace.”

The king sent messengers with gifts to the saint’s home. Upon learning of the king’s sorrow, he smiled and said, “Just as a river carries a blade of grass to the vast

ocean, so too can knowledge—even if possessed by a simple man—lead him to the king.”

Vishnusharma was received with honors at the ruler’s palace.

“I have heard that you are a very learned man, and so I wish to ask you to become a guru to my sons. However, I must warn you that no one has yet succeeded in teaching them anything. I fear they are not entirely suited to learning.”

“Just as glass is not found in a topaz deposit, so too are useless and unfit sons not born into a royal dynasty,” the Brahmin smiled.

“But how do you plan to teach them? They have no interest in learning at all!”

“I will tell them stories.”

“Stories.”

“The patterns on a clay pot cannot be changed after firing, but the impressions left in the minds of young students last a lifetime. Therefore, I will teach them by telling stories and legends.”

“Well, if you can pass on your knowledge to them, I will reward you generously.”

To which the sage replied:

– I see that you are a wise man and value knowledge. You are right in this; knowledge is the greatest wealth, because it cannot be stolen. It is priceless and imperishable but. However, I do not sell my knowledge, for I am not a merchant and did not come here for that. As an old man, I have no use for money. For six months, I will teach your sons for free.

Raja Sudarsana was delighted by the teacher’s nobility and entrusted his three sons to his care. The upper

terrace on the palace roof was set aside for their lessons. And so, at the appointed time, the princes arrived for their first lesson.

“Oh, Brahmin,” said one of them, “we heard that you were going to teach us? But we just can’t figure out why we need this.”

“We have everything: wealth, a palace, servants,” added his brother. “So why do we need your lessons?”

The teacher, hiding a smile in his beard, replied:

“I don’t think I can teach you anything; you already know so much. I’ve come simply to tell you some amusing stories. As for knowledge... Perhaps you’ve heard of a king who once lived—a very proud and haughty man. Once, a holy man came to his palace, and the king asked what was the point of his travels, since he would never amass wealth that way. The holy man replied that true wealth is knowledge. The king laughed and asked if, by any chance, he knew his future, since he was so knowledgeable. The holy man replied that he did. “Well, what awaits me?” the raja asked curiously. “You will die soon, but you are so pious that the heavenly beings will take you to their realm. There you will be far richer than you are now. However, there is one problem. There will be many mosquitoes in your palace, and although you will have a fine net to keep them out, there will be a hole in it, through which they will bite you.” The king liked the prophecy, and he assured the sage that he would simply sew up the hole, to which the saint replied: “It’s a good plan, but you haven’t considered that there are no needles in paradise. Therefore, when you are dying, make sure to take a needle with you.” – “But how can I take a needle with me?” – “I have no idea,” replied the saint. “But if you can’t even

take a needle with you, why are you so proud of your earthly wealth?"

The princes exchanged puzzled glances.

"There is something more valuable than money. Knowledge surpasses material wealth," continued Vishnusarma, "for it is the means by which all else is acquired, and when you give it away, it only increases. It is not heavy to carry, and no one can take it from you by force."

"There's some sense to that," said the third prince, settling himself more comfortably.

"Tell us something else," asked the second.

"We'll listen," agreed the first.

Next, the sage proposed a riddle to begin with, one that at first glance seemed unsolvable. And when the princes were convinced that there seemed to be no answer, he offered such a simple and elegant solution that they froze in admiration, but immediately began to insist that they would have figured it out themselves. Then he proposed another problem, about which the princes said that it certainly had no solution. But after the teacher's explanations, the solution again turned out to be so simple and elegant that the princes bit their elbows in frustration... They began to demand another problem, insisting that this time they would definitely solve it. But the wise teacher suggested that they first write down the previous solutions so they wouldn't forget them. That's how they began to improve their writing and, through that, their studies. Six months later, the instructive stories they had written down with the teacher's help made up five books—the Panchatantra.

You absolutely must have it at home.
So, where is our child?

51. How do you describe the position of a body in space?

– Where is this child walking at such a late hour?

This is a question parents ask very often. True, not every child can answer it. Let's help them with that.

First of all, let's note that this question coincides exactly with the main task of mechanics: describing the position of a body in space at any given moment. So, this is a serious question, and you shouldn't be too hard on your children's confusing answers.

To start, let's take a simpler problem. Ask your child to explain where one of their friends lives. They will most likely give their address; for example, 5 Stroiteley Street, apartment 17. But you can say that this address doesn't mean anything to you, since you don't know where Stroiteley Street is. The child will start waving their arms and explaining that you have to drive this way and that way...

Here, you can show off your erudition and mention that there was an ancient Chinese sage named Confucius who once said: if you're pointing the way, don't use your hand! And then, with an air of importance, explain Confucius's idea: humans are given language to describe what needs to be communicated; while speechless creatures or the uneducated wave their limbs about. Then, before they can catch their breath, ask:

“Do you go to the movies?”

“Of course I do.”

“And how do you find your seat in the theater?”

“By the ticket.”

“And how does the ticket work?”

– It says which row and which seat.

– Let’s say the movie has already started. It’s dark in the theater. How do you find your seat?

– There’s an usher with a flashlight; he’ll show you...

Really: why learn geography if the coachman will take you there!

But still? Let’s say row 8, seat 10. Where do you start counting the rows? Probably from something visible from anywhere in the auditorium. And what’s visible from every spot in the auditorium? The screen, of course! So we’ll count eight rows from the screen.

And the seat? We read and count from left to right. Therefore, looking from the screen, we count ten seats from left to right... There it is—ours! But someone is already sitting there... What does that mean?

– It means someone has taken my seat. We need to ask them to move...

– And he shows me his ticket, which also says: 8th row, 10th seat.

– So, they messed up at the ticket counter.

– What should we do?

– Call the usher.

– The usher looked at both tickets and said the ticket office didn’t make a mistake.

– But that doesn’t happen!

– It does. As far as I remember, there’s something else written on the ticket.

– The time!

– That’s right. One of you came to the wrong showing.

So, here’s what you need to know: where to start, where to go, how far, and... when! Therefore, if we’re visiting our friend for the first time, he needs to tell us all four details: where to start, where to go, how far, and when. When? That’s easy to figure out—just mention the time. But how do we determine from where, to where, and how far? Just like in a movie theater: we need to find a landmark whose location is known to everyone. Let’s say the Central Department Store (TSUM) is the starting point. Straight east from TSUM—that’s the direction. Walk two blocks—that’s the distance.

So, the position of a body in space can only be determined relative to other bodies whose positions are known. This is where the corresponding term comes in: – *coordinate* . However, few students can immediately define the term “coordinate.” Most students mistakenly think of a coordinate as a point. Probably because a coordinate is usually denoted by a single letter, such as x . Let’s figure this out.

We need to describe the location of a house so that the person we’re talking to understands where it is and can find it. For example, let’s say the house is four blocks east of TSUM. Here, TSUM is the reference point. East is a known direction. And a block is a unit of distance. Graphically, this can be represented as follows:

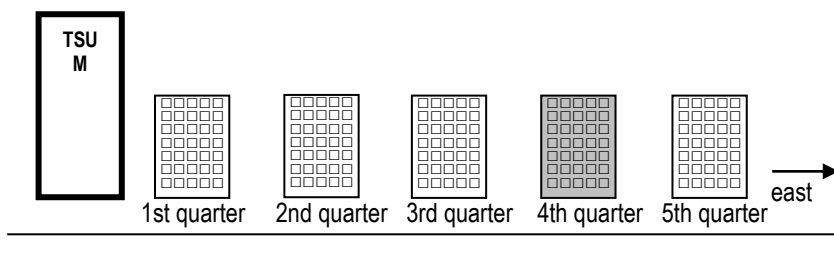


Рис. 61

The description of the house's location in this system would be: the 4th block east of TSUM.

If we simply say that the house is located four blocks from TSUM, it won't be easy to find the house right away. After all, there are four houses four blocks from TSUM: to the north, east, south, and west. But while it's fine to say in everyday speech that the house is four blocks east of TSUM, we need a universal model that meets the requirements of economy and precision. We need a *definition of a “ ”*—a supremely concise description.

52. What is a coordinate?

Let's denote the origin by the letter "O." Let's draw a directed line passing through this point and denote it by the letter "X." The directed line will be called the OX axis. Let's divide it into equal segments with a scale of one division per 1 meter. We have obtained the simplest system for describing the position of anything in space (рис. 62).

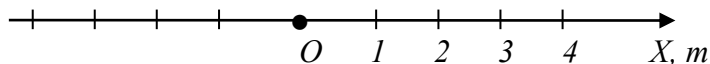


Рис. 62

So, we know: the origin, the direction of measurement, and the unit of measurement (scale).

If we receive information that a certain object (body) is located 3 meters from the origin in the OX direction, then we know where it is! (рис. 63).

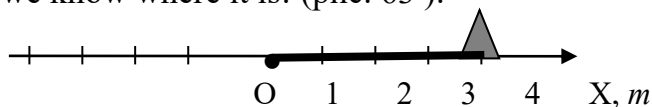


Рис. 63

We have obtained a line segment on the axis between the origin and the position of the body. Its length is 3 m. Each time, the position of the body will be determined by a similar line segment. In general, *the* length of the line segment is .

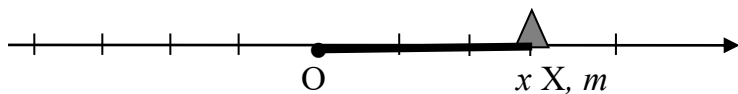


Рис. 64

The value of x is the coordinate. But the origin is always the same. Based on the principle of economy, we can omit the first letter "x" and denote the length of the segment with a single letter x .

Consequently, the position of the body in this case can be denoted by a single parameter, x , and it is equal to 3 meters.

Suppose another body is also located 3 meters from the origin, but in the opposite direction. We will denote the count in the opposite direction with the "-" (minus) sign.

Since we now have two bodies, we will denote the coordinate of the first as x_1 and the second as x_2 .

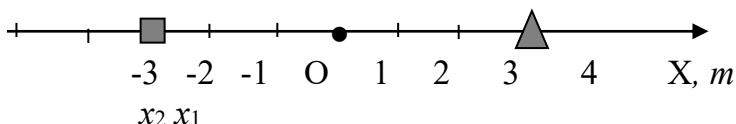


Рис. 65

In this case, $x_1 = 3\text{ m}$, $x_2 = -3\text{ m}$.

Let's note once again that the coordinate is not the point x , but what? The segment ox ?

Not exactly. In the second case, we obtained a negative value for $x_2 = -3\text{ m}$. There are no negative line segments! The distance from the origin to a point on the coordinate axis is called *the abscissa*. The abscissa can be either positive or negative.

Therefore, the coordinate of a point on a line is its abscissa.

So, knowing the body's coordinates, we know where to start measuring, in which direction to measure, and how

far. To help us do this, we use a reference point and a directed line divided into equal segments according to the chosen scale—a *coordinate system*. If we add a time scale to this, we get a *reference frame*.

Now we can specify where a given body is at a given moment in time. The parameter describing the position of a body in space is called a *coordinate*?

If we've been told—or have told someone—in which direction and at what distance from a known location an object is, isn't that enough? Why introduce coordinates and axes?

In fact, if we know where from, where to, and how far, it seems like nothing else is needed. But how do we describe how far and where to if the object is not on an axis (in a known direction)?

Let's ask again: how are the locations of various cities, countries, and continents determined? Of course, with the help of maps. Geographical maps are the greatest invention. Maps allow us to calculate distances and directions that would otherwise have to be measured directly on the ground. And that isn't always possible.

We constructed the simplest map in the previous lesson. But that was for the case where there is only one street in a village... What if there are ten?

In fact, what should we do if an object is located off to the side of a known direction?

First of all, note that a body whose dimensions can be neglected under these conditions is called a *material point* or simply a point.

Let point A lie off the OX axis (рис. 66). How do we describe its position, how do we calculate its coordinates?

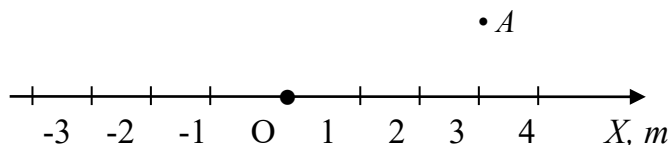


Рис. 66

We know where to start counting: the origin is “O.” We know one direction—OX. But the direction to point A does not coincide with the direction of OX. What should we do?

Option 1. Let’s construct another axis with its origin at point “O,” but directed perpendicular to the first axis. Let this be the OU axis (рис. 67).

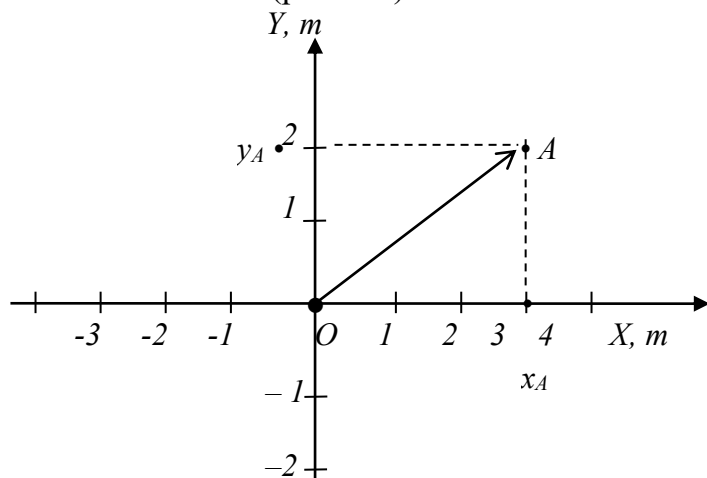


Рис. 67

To reach point A, one must follow the directed segment OA. But we do not know this direction. We know the direction of OX and the direction of OU.

Every point on the plane will have its own directed segment starting at point O. Like radii, they originate from a common center and are called radius vectors. Let’s drop

perpendiculars from the start and end of the radius vector to the OX axis. The points of intersection of the perpendiculars and the OX axis are called the projections of the corresponding points (the start and end of the vector). The segment on the OX axis between the projections of the start and end of the radius vector is called the projection of the radius vector onto the OX axis.

Unlike a line segment, a projection can be either positive or negative, depending on the direction of the radius vector itself. Consequently, *the coordinates of a point are defined as the values of the projections of its radius vectors onto the corresponding axes.*

In our case, x_A is the projection of the radius vector OA onto the x-axis. Therefore, x_A is the x-coordinate of point A. We have already called this *the absciss*; y_A is the y-coordinate of point A. This will be called *the ordinate*.

Accordingly, $x_A = 3\text{ m}$, $y_A = 2\text{ m}$. The coordinates of point A are written as follows: A (x;y). In our case, A (3;2).

Now, let's try to solve the problem.

Problem 1. In a rectangular coordinate system, calculate the distance between two points A(1;2) and B(4;6) without using a ruler.

Of course, if we had a ruler, measuring the length of segment AB would be easy. But according to the problem's conditions, we cannot use a ruler. Let's use the coordinate system and our previous lessons. First, we'll draw a diagram that includes all the terms we know: coordinates, the origin, coordinate axes, and projections of points.

We need to calculate the distance between points A and B without using a ruler,

Let's draw a perpendicular line dropped from point A to the OY axis to the right until it intersects with the perpendicular line dropped from point B to the OX axis. Let's label the point of intersection with the letter C.

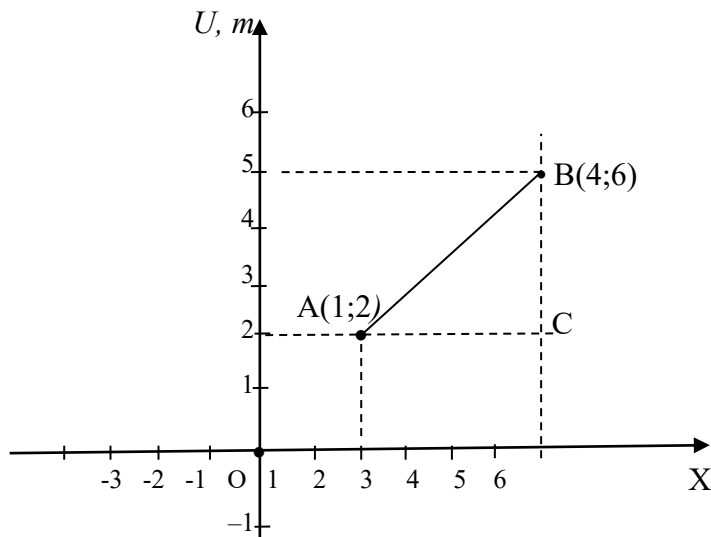


Рис. 68

We have obtained a right triangle ABC.

In triangle ABC, we know the lengths of the legs AC and BC. They are 3 *m* and 4 *m*, respectively.

By the Pythagorean theorem,

$$AB^2 = AC^2 + BC^2,$$

$$AB = \sqrt{AC^2 + BC^2}.$$

$$\text{Therefore, } AB = \sqrt{3^2 + 4^2} = 5 \text{ (} m \text{)}.$$

The problem is solved.

But this is a graphical method of solution. Is it possible to solve it without drawing a diagram?

From the figure, it is clear that the length of the side AC is equal to the difference between the coordinates x_B and x_A .

Therefore, $AC = x_B - x_A$.

Similarly, for the leg BC:

$BC = y_B - y_A$.

Now we can write down a general formula for calculating the distance between two points with known coordinates:

$$AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}.$$

So, if we know the coordinates of two points, we can *calculate* the distance between them. There are two ways to do this: graphically and algebraically.

Conclusion: A coordinate system allows us to calculate parameters that are difficult or impossible to measure using parameters that are easy to measure.

Option 2. Let's connect the origin to point A with a line segment. We can measure the length of this segment and the angle between it and the x-axis.

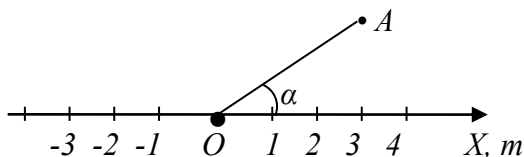


Рис. 69

Here, the position of the body can be described as follows: point A is located at a distance OA from the origin at an angle α to the known direction OX.

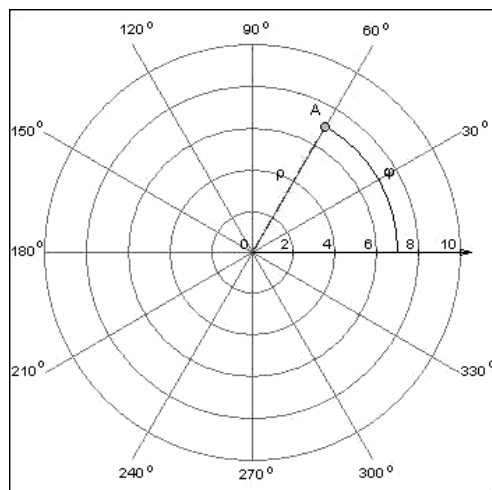


Рис. 70

Here, the origin “O” is called the pole. The position of a body is determined by the distance from the pole to the point and the angle between the known direction and the direction to the point, measured counterclockwise.

This coordinate system is called *polar*.

So, we have two ways to describe the position of a body in space: the first is using a rectangular (or Cartesian) coordinate system; the second is using a polar coordinate system.

But what if the body’s coordinates change?

53. What is a change in magnitude?

How do we describe the position of a body if its coordinates change?

After asking this question, ask your child if all the words in it are clear.

"Description" is familiar to them. "Coordinate" — too! They will confidently say that everything is clear.

Now ask, what is a change ?

– Well, a change is when something changes...

– We have a rule: a definition must not contain a word that shares the same root as the term being defined .

–...?

– You had five apples; now you have seven. What is the change in the number of apples?

– Two!

– Correct. You had seven apples, now you have five. What is the change in the number of apples?

– Two!

– Two again? Okay. The number of apples changed by two. Are there more or fewer now?

– Well, in the first case, there are more, and in the second, there are fewer.

– But what if we don't know the previous cases? For example, you were told that your bank account has changed by one million rubles. Is there more money or less? Should you be happy or sad?

–...?

Changes occur only over time. Time has only one direction. That is why the phrases "was" and "became" make sense to us.

The change in a quantity is defined as the difference between its subsequent and initial values.

It was five, it became seven. Here, the initial value is five, the subsequent value is seven. Therefore:

$$7 - 5 = 2.$$

The change in is equal to two.

It was seven, now it is five. Here, the initial value of is seven, and the subsequent value is five. Therefore:

$$5 - 7 = -2.$$

The change in is equal to minus two.

If the change in is positive, then the value of increases; if it is negative, then it decreases.

Therefore, the balance in the bank has increased by one million. After all, the change is positive.

The change in the parameter is the difference (difference). Therefore, the change in the parameter is denoted by the same letter as the parameter, only preceded by the Greek letter delta (Δ).

For example, the change in the x -coordinate is Δx .

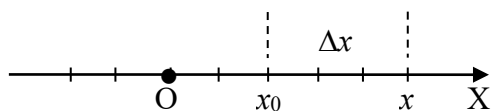


Рис. 71

If x_0 is the initial coordinate of and x is the final coordinate, then the change in the coordinate is equal to:

$$\Delta x = x - x_0.$$

The task is to calculate the final coordinate at any given time.

Let us assume that over equal time intervals, the coordinate changes by the same amount.

Suppose that in 1 second, the coordinate changes by 5 meters. Then, it is easy to calculate what the coordinate will be after, say, seven seconds. It is 35 m.

Let's denote the change in the coordinate per unit of time by the letter v , and the time interval by the letter t .

Then, the change in the coordinate over any time interval will be:

$$\Delta x = v \cdot t.$$

The change in a quantity per unit of time is called the rate of change of that quantity, or simply *velocity*. Velocity, in this context, is a term that denotes the relationship between two other terms. Such terms are called *concepts*. Consequently, velocity is a concept. The quantity of velocity is denoted by the letter v —the first letter of *the* English word “velocity.”

Δx – letters denoting the magnitude of the change in coordinate, i.e., this is the parameter. It can be measured

t – a letter denoting the time interval, i.e., this is a parameter. It can be measured.

And what is v ? It is also a parameter of, because it is a letter... But what quantity does it denote?

It denotes the value of one parameter per unit of another. Such parameters are called **specific values** or **coefficients**.

If a body moves uniformly and in a straight line along the OX axis, then the change in the coordinate is equal to the displacement of the body. Let's denote the displacement by the letter s . Then, $s = \Delta x$. Therefore,

$$s = v \cdot t.$$

We have already encountered this formula when solving motion problems. Here, v is the coefficient that relates the parameters s and t . It shows how much s there is per unit of t , i.e., what displacement the body undergoes per unit of time.

For example, if a car's speed is 60 km/h , this means it travels 60 km in 1 hour . Knowing the distance it travels in 1 hour, we can calculate the distance it will travel in 2 hours, 5 hours, and so on.

$s = v \cdot t$ is the basic formula. From it, we can derive formulas for calculating speed and time. To do this, we simply set up this formula as an equation and solve it first for v . To do this, we divide both sides of the equation by t .

$$s = \frac{v \cdot t}{t} \Rightarrow s/t = v$$

Now let's solve it for t . To do this, we'll divide both sides by v .

$$\frac{s}{v} = \frac{v \cdot t}{v} \Rightarrow s/v = t$$

To calculate the average speed of motion, simply divide the total distance traveled by the total time of motion. This gives us the distance traveled per unit of time—a specific quantity that helps us in our calculations.

54. What is a function?

In a conversation with a student, drop the phrase: “That’s not my function !” or “Do you have an argument?” After receiving any answer, ask, “What is a function?” and “What is an argument?”

We’re familiar with parameters, formulas, and equations. It’s time to move on to functions. Especially since we’ve been using them for a long time. We just need to distinguish them, describe them, and formulate a definition .

At first, children are introduced to parameters. But for them, parameters are constant values that do not change. For example, the parameters in our first formula for the circumference, $c = 2\pi r$. In each specific case, the parameters c and r have only one value each; that is, for a given circle, these values are constant.

But we have already encountered changing quantities. For example, the change in the coordinate over time.

Parameters denoting changing quantities are called **variable parameters** or simply **variables**.

In this context, we distinguish between **dependent and independent variables**.

For example, displacement depends on time.

Mathematically, this is written as:

$$s = vt$$

Here, t (the value of time) is the independent variable; s (the value of displacement) is the dependent variable ; v (the speed of displacement) is the coefficient of dependence.

The dependent variable is called **the function** , and the independent variable is called **the argument**.

The relationship between the function and the argument is generally written as follows:

$$y = f(x)$$

Consequently, in general form, the relationship between displacement and time is written as:

$$s = f(t)$$

We say that displacement (s) is a function of time (t). Time (t) is the argument here.

Any formula we already know, such as , can be expressed as a function by replacing the constant parameters with variables.

For example, the circumference depends on the radius:

$$c = f(r);$$

the area of a square is a function of the length of its side

$$S = f(a).$$

A graph of a function clearly demonstrates how one variable changes as another changes.

Let's plot the graph of the function $y = 2x + 3$.

Since the variable x is the argument, i.e., the independent variable, we can assign any value to it. And the variable y is the function $y = f(x)$, i.e., the dependent variable; its value will depend on the value assigned to the argument x .

For example,

$$\text{if } x = 0, \text{ then } y = 2 \cdot 0 + 3 = 3;$$

$$\text{if } x = 1, \text{ then } y = 2 \cdot 1 + 3 = 5;$$

$$\text{if } x = 2, \text{ then } y = 2 \cdot 2 + 3 = 7;$$

if $x = 3$, then $y = 2 \cdot 3 + 3 = 9$;

if $x = 4$, then $y = 2 \cdot 4 + 3 = 11$;

and so on.

Let's construct a coordinate system. Let's find all these points.

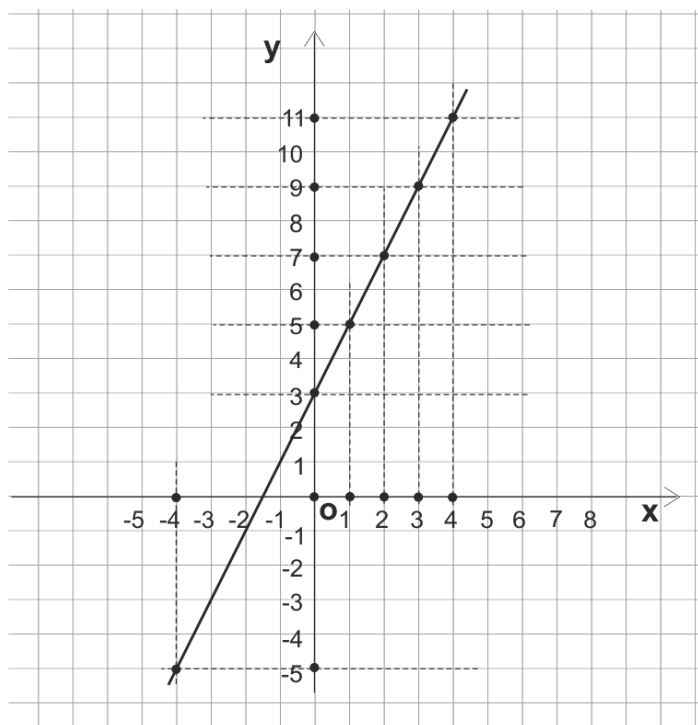


Рис. 72

We know that if $x = 0$, then $y = 3$. Let's mark these two points on *the x-* and *y-axes*, respectively. Let's draw the perpendiculars to the axes from these points. Let's mark the point where they intersect. In our case, the point of intersection lies directly on *the y-axis*.

If $x = 1$, then $y = 5$. Let's mark these points and draw the perpendiculars to them. Let's mark the point of intersection of these perpendiculars.

Similarly, mark the positions of the points for the cases where $x = 2$, $x = 3$, and $x = 4$.

Now let's connect the resulting points of intersection of the perpendiculars. We get a straight line.

It is easy to see that all points lying on this line correspond to the function $y = 2x + 3$.

For example, let's mark the point $x = -4$ on the graph. Draw a perpendicular from this point to *the x-axis*. From the point of intersection of this perpendicular and our line, draw a perpendicular to *the y-axis*. The point of intersection of this perpendicular and *the y-axis* will be (-5) . It corresponds to the value of the function calculated using the formula:

$$y = 2x + 3 = 2(-4) + 3 = -5.$$

Therefore, a function whose independent variable is raised to the first power is called a linear function .

Here, x is raised to the first power. As we know, the first power is not written out, since $x^1 = x$. In general form, the linear function is written as

$$y = kx + b,$$

where k and b are any numbers.

Let's plot the graph of for another function:

$$y = x^2$$

Here,

if $x = 0$, then $y = 0^2 = 0$;

if $x = 1$, then $y = 1^2 = 1$;

if $x = 2$, then $y = 2^2 = 4$;

if $x = 3$, then $y = 3^2 = 9$;

For convenience, let's construct a table of values of the independent variables and their corresponding function values.

x	0	1	2	3	...
at	0	1	4	9	...

Let's construct a coordinate system, mark the corresponding points, and connect them with a smooth curve

Now let's assign negative values to the argument (the independent variable).

$$y = x^2$$

if $x = -1$, then $y = (-1)^2 = 1$;

if $x = -2$, then $y = (-2)^2 = 4$;

if $x = -3$, then $y = (-3)^2 = 9$;

x	-1	-2	-3	...
at	1	4	9	...

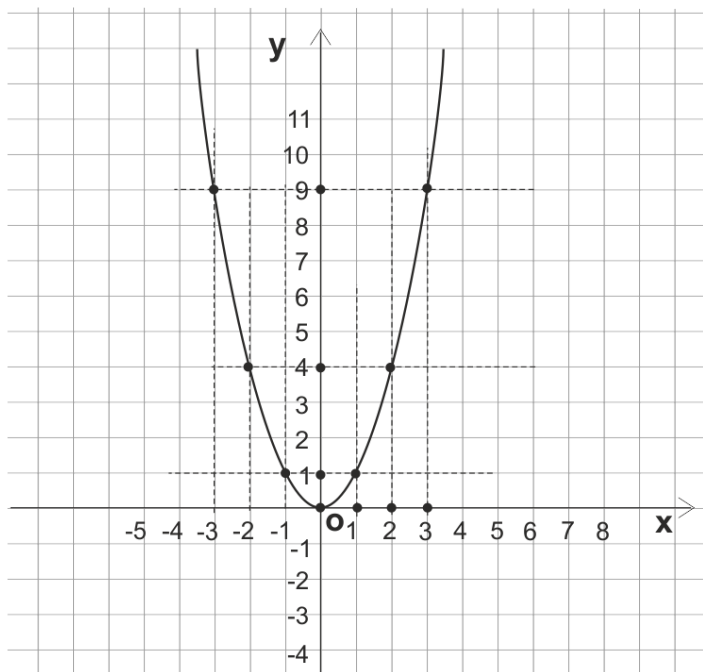


Рис. 73

This is the graph of a quadratic function. The resulting curve is called a **parabola**. The branches of the parabola are symmetric about a line passing through its vertex and parallel to the *y*-axis.

In general form, the quadratic function is written as follows:

$$y = ax^2 + bx + c,$$

where a , b , and c are any numbers, but $a \neq 0$.

When $a = 0$, the quadratic function becomes linear.

55. What does it mean to analyze a function?

So, a function is a parameter, the value of which depends on another parameter, the argument. A description of exactly how the function changes is called an investigation or analysis of the function.

Let's try to investigate the variation of the function using its graph.

Suppose we already have the graph of the function $y=f(x)$. Рис. 74 .

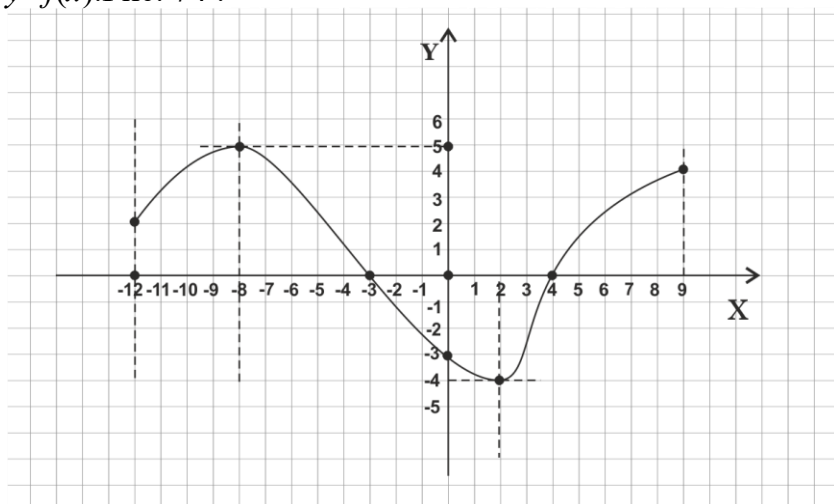


Рис. 74

Will your child be able to analyze this function? He or she will most likely say no.

Now let's list a series of questions that need to be answered to construct an analysis. They will be able to answer each of them individually.

1. What is the domain of the function? That is, within what range of values of the argument x is the function y defined?

(We look at the values of x on the x -axis. Within the range from $x=-12$ to $x=9$, or the domain of the function: $x \in [-12;9]$).

2. What is the range of the function? That is, within what limits do the values of the function itself lie?

(We look at the values of the function on the OY -axis: in the interval from $y = -4$ to $y = 5$; the range of the function: $y \in [-4;5]$).

3. What are the zeros of the function? At which points does the graph of the function intersect the OX -axis? That is, for which values of the argument x does the function become zero?

(Here, $x_1 = -3$ and $x_2 = 4$)

4. At which points does the graph of intersect the OY -axis? What are the values of the function when the argument is zero, i.e., when $x=0$?

(Here, this is the point with coordinates $y = -3$).

5. What is the maximum value of the function?

($y_{\max} = 5$).

6. What is the minimum value of the function?

($y_{\min} = -4$).

7. In which intervals of the argument's values does the function not change its sign?

(There are three such intervals: $(-12;-3)$, $(-3;4)$, $(4;9)$).

8. In which intervals of the argument x does the function y increase?

(There are two such intervals: $(-12;-8)$, $(2;9)$)

9. In which intervals of the argument x does the function y decrease?

(There is one such interval: $(-8;2)$).

56. Solving inequalities using the interval method

A change in the argument within a given interval of values changes the values of the function. The values of the function (the values of y) can be positive, negative, or zero. Moreover, the sign of the function changes at the point $y=0$. And if the function is continuous, then after a change in sign, for example, from negative to positive, the next change can only be from positive to negative.

Consider the graph of some function $y=f(x)$ defined on the interval $[x_1; x_5]$. Рис. 75 .

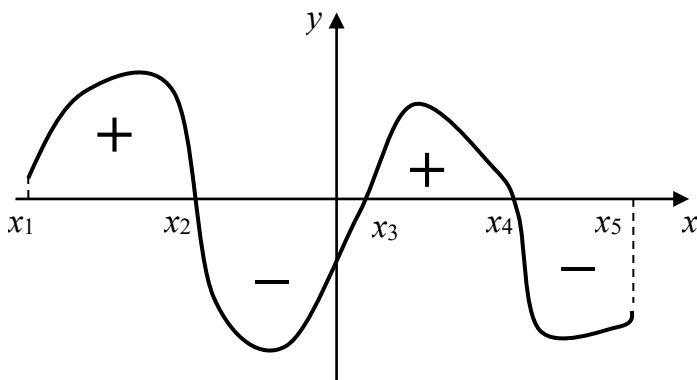


Рис. 75

Here, in the interval of argument values from x_1 to x_2 , the function is positive ($y > 0$). Next, at the point x_2 , it changes sign from positive to negative and remains negative until the argument reaches x_3 . At the point x_3 , the function changes sign again, this time from negative to positive, and remains positive until the argument reaches x_4 ().

At the point x_4 , the function changes sign from positive to negative.

At the points x_2 , x_3 , and x_4 , where the function changes sign, the value of the function itself is zero.

This fact will help us continue solving the inequalities we started in one of the previous lessons.

Consider the inequality :

$$(x-7)(x+3)(x-4) > 0$$

To find its solution, let's express the inequality as a function:

$$y = (x-7)(x+3)(x-4) > 0$$

For what values of the argument x does *the* function become zero?

These values are: $x_1 = 7$, $x_2 = -3$, $x_3 = 4$.

It is precisely at these points that the graph of the function intersects the x -axis, and, accordingly, the function changes sign. Let's mark these points on the x -axis. There is no need to draw the y -axis, since we do not need specific values of the function y . We only need the intervals of the argument x in which the function takes values greater than zero.

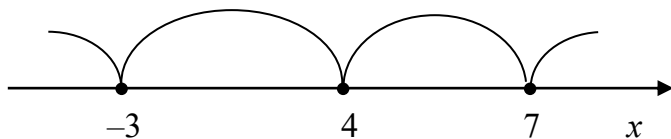


Рис. 76

The graph shows four intervals. But what are the signs of the function in each interval?

Let's consider the rightmost interval. Here, the values of the argument are greater than 7. If we substitute

values of the argument x greater than 7 into the formula for our function, the expressions in all three parentheses are positive. Therefore, in the rightmost interval, the values of the function will be positive. This means that from right to left in each interval, the sign will change to the opposite. The points themselves do not fall within the interval, since we have a strict inequality. Therefore, we will denote them with hollow circles.

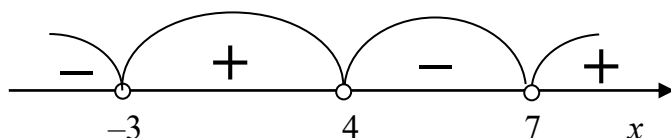


Рис. 77

The solution to our inequality

$$(x-7)(x+3)(x-4) > 0,$$

we get

$$x \in (-3; 4) \cup (7; +\infty).$$

Here, the interval is bounded by round brackets, which means that the endpoints themselves are not included.

If the inequality were not strict, the circles would be solid, and the brackets would be square.

If $(x-7)(x+3)(x-4) \geq 0$, then $x \in [-3; 4] \cup [7; +\infty)$.

This method of solving inequalities is called *the interval method*.

57. What is the sine of an angle?

– Is it possible to calculate the length of a leg if the length of the hypotenuse is known?

– Of course! Using the Pythagorean theorem.

– Okay. Let the length of the hypotenuse be 10 *cm*.

What is the length of the leg?

– And what is the length of the other leg?

– We don't know.

– Then, the length of the hypotenuse cannot be calculated.

– What if the angle between the leg and the hypotenuse is known?

– Well, I don't know!

– And what is the angle between the legs?

– 90° .

– And what is the sum of the other two acute angles?

– Also 90° .

– Why?

– Because the sum of the angles in a triangle is 180° .

– That needs to be proven.

– How? I've already forgotten!

– Let's construct a triangle. Draw a line through one of its vertices that's parallel to the base. Let's label the resulting angles. What are they called?

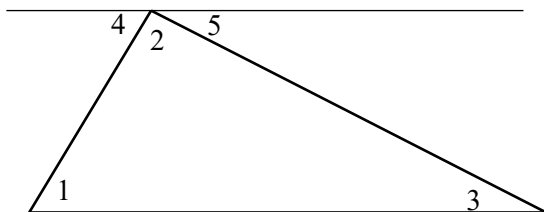


Рис. 78

– Angles 4 and 1 are alternate interior angles. They are equal. Angles 5 and 3 are also equal.

– What is the sum of the angles 4, 2, and 5 ()?

– 180° .

– Great! And what is the sum of angles 1, 2, and 3?

– ?

– In the first sum, replace angles 4 and 5 with angles 1 and 3, which are equal to them.

– It will also be 180° .

– Proven!

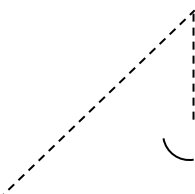
In a right triangle, one angle is always equal to 90° . Therefore, the sum of the other two angles is equal to:

$$\angle \alpha + \angle \beta = 90^\circ$$

Changing one of them always leads to a change in the other. For this to hold true, the lengths of the legs and the hypotenuse must also change. In turn, changing the length of any of the legs of a right triangle leads to a change in the acute angles (α and β) and the length of the hypotenuse.

Let's examine each case in more detail.

Let the length of side b remain constant. Then, any change in the value of angle α will lead to a change in the lengths of side a and the hypotenuse c . (Рис. 79).



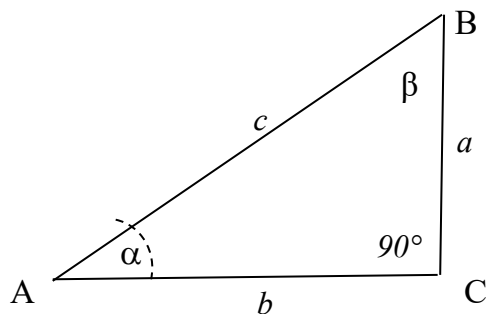


Рис. 79

Let the length of the leg a remain constant. Then any change in the angle α will result in a change in the lengths of the leg b and the hypotenuse c . (Рис. 80).

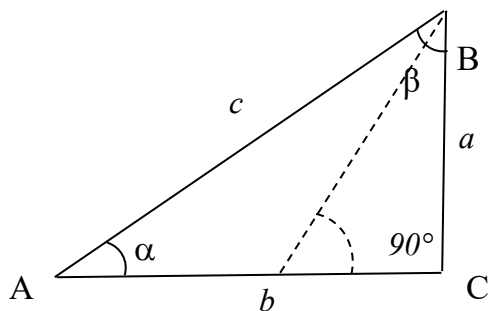


Рис. 80

In a right triangle, there is a certain relationship between an acute angle and the pair of sides consisting of the legs and the hypotenuse.

*The length of the side opposite a given acute angle, relative to one unit of the hypotenuse, is called **the sine** of the angle.*

To calculate the sine of an angle, divide the length of the side opposite that angle by the length of the hypotenuse. Their ratio is $\sin \alpha$.

$$\sin \alpha = \frac{a}{c}$$

Consequently, the sine of an angle is a ratio, i.e., a value obtained by division that indicates how much of one quantity corresponds to a unit of another.

The basic sine formula looks like this:

$$a = (\sin \alpha) \cdot c$$

This formula is analogous to the well-known formula for speed:

$$s = v \cdot t$$

Here, knowing the distance per unit of time (v), we can calculate the distance (s) traveled over any time period (t).

But where do we get the sine values of the angles?

Today, you can simply use a good calculator, enter the angle, press the sin function, and immediately get the length of the side opposite the angle per unit length of the hypotenuse. In the old days, special tables were used for this.

The cosine of an angle, respectively, is the length of the adjacent side per unit length of the hypotenuse. To calculate the cosine of an angle, you must divide the length of the adjacent side by the length of the hypotenuse. Their ratio is $\cos \alpha$.

$$\cos \alpha = \frac{b}{c}$$

58. How do you calculate the sine values for 30, 45, and 60 degrees?

Enough theory. Now let's do some problems. I wonder what the sine of is for a 30° angle.

It's clear that the kids aren't interested in this at all.

But you'll tell them that they have everything they need to solve this problem. But will they be able to solve it, even with everything they need? This is a matter of principle, a competition, a test of their ability. Here, failure will be seen as a defeat. And kids don't want to lose.

A quick-witted child will say that they need to take a calculator, type in 30, and press "sin." That's all correct. But what will the knowledge that the calculator will show 0.5 give this perceptive child? Does he understand what this answer means and how it is derived in practice? And will he be able to pose the problem to the calculator on his own next time?

So many questions. We can't wait too long for an answer. We need to help the child in time. We need to suggest a slightly different game: you quickly show them once how to solve the problem, and they have to repeat it. If they succeed, they've won the first stage. Next time, with that experience, they should be able to do it all on their own.

So, let's demonstrate in practice what the sine of thirty degrees is. How do you solve this problem?

Just like any other: you need to boldly do what you know how to do and not do what you don't know how to do! The main thing here is to boldly start acting without hesitation.

So, let's calculate the value of the sine of 30° .

We know that for any acute angle in a right-angled triangle

$$\sin \alpha = \frac{a}{c}$$

Therefore,

$$\sin 30^\circ = \frac{a}{c}$$

Now, let's draw a right triangle, label all the parts, and introduce variables—letters representing quantities. After all, it is the relationships between these quantities that we are looking for. We know how to do this.

Let the lengths of the legs be a and b , respectively. The length of the hypotenuse is c . Aside from the Pythagorean theorem and the sine formula, no other relationships between the values are apparent here. This is not enough for now. How can we increase the number of relationships?

Let's extend the triangle into a right triangle and draw a second diagonal in it.

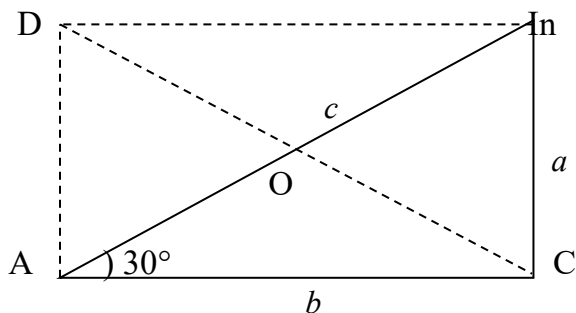


Рис. 81

Let's denote the point of intersection of the diagonals by the letter O. Now we have many more relationships between the measures of the angles and the lengths of the sides.

If one acute angle in a right triangle is 30° , then the other must be 60° (since their sum must equal 90°).

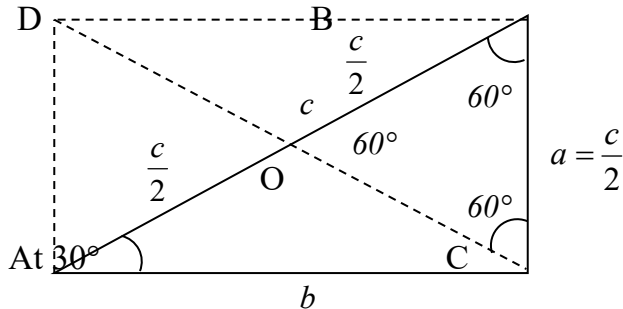


Рис. 82

The diagonals of a rectangle are equal to each other and bisect each other when they intersect. Therefore, segment OB is equal to segment OC. Consequently, triangle OBC with base BC is isosceles. And in an isosceles triangle, the angles at the base are equal. Therefore, angle OSB is equal to angle OBC and is 60° .

But if two angles of a triangle are 60° , then the third angle must also be 60° so that the sum of all angles is 180° .

If all three angles of a triangle are equal to each other, then all three of its sides are also equal to each other. Therefore, $BC = CO = OB$.

But, $BC = a$, and $OB = \frac{c}{2}$.

Therefore, $a = \frac{c}{2}$.

Let's substitute this value into the sine formula.

$$\sin 30^\circ = \frac{a}{c} = \frac{\frac{c}{2}}{c} = \frac{1}{2}$$

Thus, $\sin 30^\circ = \frac{1}{2}$.

Therefore, if in a right triangle a leg is equal to half the hypotenuse, then opposite this leg is an angle of , which is 30° . And vice versa.

Go through the entire derivation process with your child one more time. Then, give them a blank sheet of paper and have them try to repeat it. If they don't get it right the first time, let them look at their previous notes. Make sure they can derive the entire result on their own.

At the same time, remind them that we simply did what we know how to do and didn't do what we don't know how to do. Therefore, any excuses that they can't do this are not acceptable.

Now, let him try to calculate the sine of 60° on his own.

Clearly, they will figure out how to extend the right triangle into a rectangle. But then they will hit a dead end. They will find that half of the hypotenuse is not equal to side a , but to side b .

In fact, using the same approach as in the previous problem, we find that triangle AOB is now equilateral. Therefore, the length of side b is equal to half the length of the hypotenuse.

But we need side a , not side b .

What should we do?

As soon as the terms “length of the hypotenuse” and “length of the leg” are mentioned, we immediately apply the Pythagorean theorem to this triangle.

Will it help?

We don’t know. But we know how to do it, so we go ahead and do it. There’s nothing else we can do.

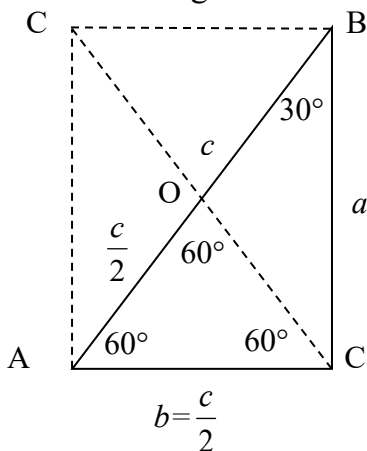


Рис. 83

According to the Pythagorean theorem:

$$a^2 + b^2 = c^2$$

Substituting $b = \frac{c}{2}$, we get

$$a^2 + \left(\frac{c}{2}\right)^2 = c^2 \quad \Rightarrow \quad a^2 + \frac{c^2}{4} = c^2 \quad \Rightarrow$$

$$a^2 = c^2 - \frac{c^2}{4} \quad \Rightarrow \quad a^2 = \frac{3}{4} c^2 \quad \Rightarrow$$

$$a = \sqrt{\frac{3}{4}c^2} \Rightarrow a = \frac{c}{2}\sqrt{3}.$$

Substitute the resulting value of a into the formula for the sine of a 60° angle.

$$\sin 60^\circ = \frac{a}{c} = \frac{\frac{c}{2}\sqrt{3}}{c} = \frac{\sqrt{3}}{2}$$

So, we have obtained the sine values of two standard angles:

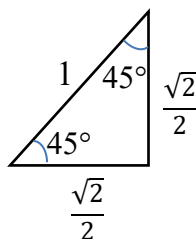
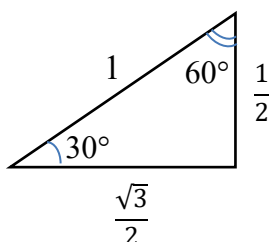
$$\sin 30^\circ = \frac{1}{2} \quad \text{and} \quad \sin 60^\circ = \frac{\sqrt{3}}{2}$$

Now, let's try to calculate the sine of a 45° angle.

Just note that in this case, there is no need to extend the triangle to form a right triangle.

The same calculations, to reinforce the material covered, can be suggested for the cosines of angles of 30 , 45 , and 60 degrees, respectively. This will result in a table that can be supplemented with a diagram:

α	30°	45°	60°
$\sin \alpha$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos \alpha$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$



59. How do you calculate the sine of 0 and 90 degrees?

I wonder, what is the sine of 0 degrees ()?

Of course, we're being a bit sneaky here: a child isn't interested in this at all. But we'll try to get them interested. Ask them to draw a right triangle with an acute angle of 0° to find out what $\sin 0^\circ$ is equal to.

The very first attempt will lead the child to conclude that such a triangle does not exist.

And you suggest the following line of reasoning: if, in a right-angled triangle, we reduce the acute angle to zero, then the opposite side will also decrease to zero.

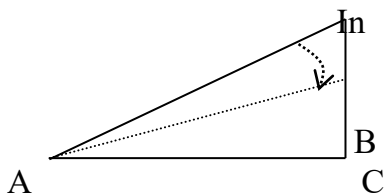


Рис. 84

This results in an unusual right triangle:

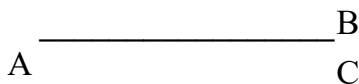


Рис. 85

If the child says that this is just a line segment, tell them that it satisfies all the criteria for a triangle. Here, angle ACB is a right angle. Angle ABC is also a right an-

gle, and angle BAC is zero. The sum of all the angles is 180° . Just as it should be for a triangle.

The length of the leg AC is equal to the length of the hypotenuse AB, and the length of the leg BC is zero.

The sine of angle BAC is equal to the ratio of the length of the opposite side (BC) to the length of the hypotenuse (AB). Since angle BAC is zero, the length of side BC is also zero, so the sine of this angle is also zero. That's the geometry.

It remains to be determined what the sine of a 90° angle is.

With the leg BC fixed, we will increase the acute angle in the right triangle to 90° . As soon as this happens, the angles at the base will become equal to each other.

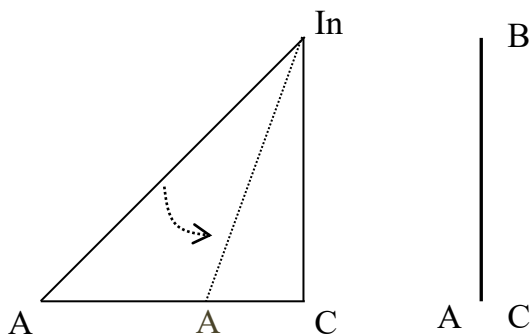


Рис. 86

We end up with another strange right-angled triangle.

Nevertheless, it is a right triangle, and an isosceles one at that; that is, the side opposite our angle will be

equal to the hypotenuse. Their ratio is 1. Therefore, the sine of for a 90° angle is 1.

Now let the child try to calculate the values of the cosines of 0° and 90° on their own. They are familiar with the line of reasoning.

It is very important here to note that we are considering a dynamic process, yet we obtain a static result. Remind them that the same approach was used in the lesson on calculating the area of a circle: the number of sides of the inscribed regular polygon tended to infinity (a dynamic process), and in the limit, its area was equal to the area of the circle (a static result).

Let's complete the table of sine and cosine values for angles:

α	0°	30°	45°	60°	90°
$\sin \alpha$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos \alpha$	1	$\frac{\sqrt{3}}{2}$		$\frac{1}{2}$	0

Have the child analyze the table of values. They are already experienced enough to notice some symmetry and draw conclusions.

Now let's use their own observations to determine the relationship between the sine and the cosine of the angles in a right triangle.

In fact, in a right triangle, the sine of one acute angle is always equal to the cosine of the other acute angle, and vice versa. This follows from the definitions of the sine and cosine of an angle. Let the child try to draw this conclusion on their own.

They should arrive at the following:

“The sine of an angle is equal to the ratio of the opposite side to the hypotenuse, and the cosine of an angle is equal to the ratio of the adjacent side to the hypotenuse.

Since in a right triangle

$$\sin \alpha = \frac{a}{c} \quad \text{and} \quad \cos \beta = \frac{a}{c} ,$$

then $\sin \alpha = \cos \beta$.

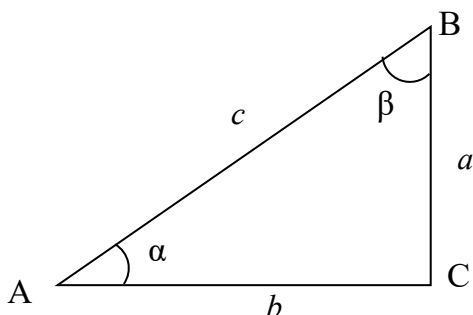


Рис. 87

If the child arrives at this conclusion, praise them and explain that this conclusion can be used to derive a wonderful formula.

Recall that in a right triangle, the sum of the of its acute angles equals a right angle.

$$\angle \alpha + \angle \beta = 90^\circ$$

Therefore,

$$\angle \beta = 90^\circ - \angle \alpha.$$

Now let's write down the formula obtained above ($\sin \alpha = \cos \beta$) and replace $\angle \beta$ with what it equals, i.e., $(90^\circ - \angle \alpha)$.

We obtain

$$\sin \alpha = \cos (90^\circ - \alpha).$$

Similarly,

$$\cos \alpha = \sin (90^\circ - \alpha).$$

Let's express the angles in radians.

We know that $90^\circ = \frac{\pi}{2}$ *radians*.

Therefore,

$$\sin \alpha = \cos \left(\frac{\pi}{2} - \alpha \right).$$

Similarly,

$$\cos \alpha = \sin \left(\frac{\pi}{2} - \alpha \right).$$

60. The relationship between the sine and cosine of a given angle

We have examined the relationships between the lengths of the legs and the hypotenuse for these angles. Similarly, we can derive relationships between the lengths of the legs and the corresponding acute angles. This is how the *tangents* and *cotangents* of the angles are introduced.

In a right triangle, the length of the opposite side per unit length of the adjacent side is called the tangent ($\operatorname{tg} \alpha$) of that angle. And the length of the adjacent side per unit length of the opposite side is called the cotangent of the angle ($\operatorname{ctg} \alpha$).

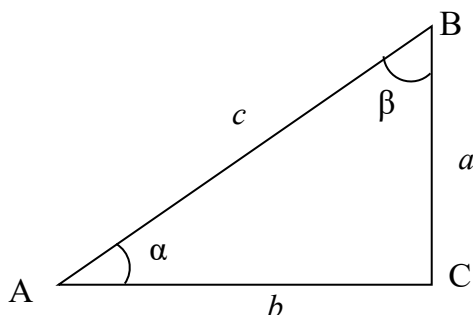


Рис. 88

Therefore,

$$\tan \alpha = \frac{a}{b} \quad \text{and} \quad \cot \alpha = \frac{b}{a} . \quad \text{Therefore, } \tan \alpha = \frac{1}{\operatorname{ctg} \alpha} .$$

Now we can solve the problems.

Have the child try to express the tangent of the angle in terms of its sine and cosine ?

Remind them again that they have all the knowledge needed to solve this problem. Will they be able to use it?

If not, remind him that he just needs to do what he can do:

1. Can you write down the formula for the tangent of an angle?

$$(\tan \alpha = \frac{a}{b})$$

2. Can you write down the formula for the sine of an angle?

$$(\sin \alpha = \frac{a}{c})$$

3. Can you calculate the side b ?

$$(a = c \cdot \sin \alpha)$$

4. Can you write down the formula for the cosine of the angle?

$$(\cos \alpha = \frac{b}{c})$$

5. Can you calculate the side b ?

$$(b = c \cdot \cos \alpha)$$

6. Can you substitute the parameters a and b into the formula for the tangent of an angle?

$$(\operatorname{tg} \alpha = \frac{a}{b} = \frac{c \cdot \sin \alpha}{c \cdot \cos \alpha} = \frac{\sin \alpha}{\cos \alpha})$$

So, the ratio of the sine of a given angle to its cosine gives the tangent of that angle.

And how did we figure out that this was exactly what we needed to do?

We didn't. We didn't know how to do anything else here. We simply did what we knew how to do.

Most likely, at some point the child will figure out on their own how to finish the solution. Even if they don't, once the proof is complete, it will be clear to them that the solution was quite simple. They might even claim that they would have figured it out on their own. That's when you give them the next problem. Let them show what they're capable of.

And here is the problem: identify the relationship between the sine of an angle in a right-angled triangle and the cosine of that angle.

For most students, such problem conditions are still not entirely clear. Therefore, let's rephrase the question: what is the relationship between the elements that make up the sine and the cosine of the angle? That is, what is the relationship between the legs and the hypotenuse? He should recall the Pythagorean theorem. Have him write it down using its parameters.

$$a^2 + b^2 = c^2$$

How do we convert all of this into sine and cosine values? We need to divide the legs by the hypotenuse. Let him not be confused by the squares of the legs. Let's divide the left and right sides of this formula by the square of the hypotenuse. We get:

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2} .$$

$$\sin^2\alpha + \cos^2\alpha = 1$$

How beautiful!

The sine of an angle can be expressed in terms of its cosine as follows:

$$\sin^2\alpha = 1 - \cos^2\alpha.$$

Therefore,

$$\sin\alpha = \sqrt{1 - \cos^2\alpha}.$$

And conversely:

$$\cos^2\alpha = 1 - \sin^2\alpha.$$

Therefore,

$$\cos\alpha = \sqrt{1 - \sin^2\alpha}$$

Now give our student a problem.

What is the sine of the angle, if its cosine is equal to $\frac{4}{5}$?

Most likely, he will start thinking. In that case, tell him that he shouldn't think. As soon as he starts thinking, lower his grade by one point. Usually, when a child can't solve a problem, they're told to think. But thinking is necessary when studying theory. When solving problems, it's already too late to think.

We immediately write $\sin\alpha = \sqrt{1 - \cos^2\alpha}$, substitute the value of the cosine, and calculate:

$$\sin\alpha = \sqrt{1 - \cos^2\alpha} = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \sqrt{1 - \frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

When you have the theory, you must act decisively.

61. The Law of Cosines

For some children, these are the most incomprehensible—and therefore the most frightening—words. But we've already learned to dispel fears. Let the child try to figure out these theorems on their own. You just help frame the problem. Besides, they're not a child anymore; they're a real student!

Let's consider an arbitrary triangle.

Two sides (a and b) and the angle (α) between them have been measured.

Ask again: how can we measure the lengths of the sides, and how can we measure the angle ?

Suppose we cannot measure the third side (c)... Can we calculate its length?

Clearly, if the angle between sides a and b were 90° , we would have a right triangle, and by the Pythagorean theorem, the length of side c could be calculated.

But here, all the angles are acute. The triangle is not a right triangle. What should we do?

As always, we'll do what we can and not do what we can't.

Let's draw the triangle.

Let's draw the altitude (h) and set the parameters. We know how to do that.

We have obtained two right triangles. (Рис. 89).

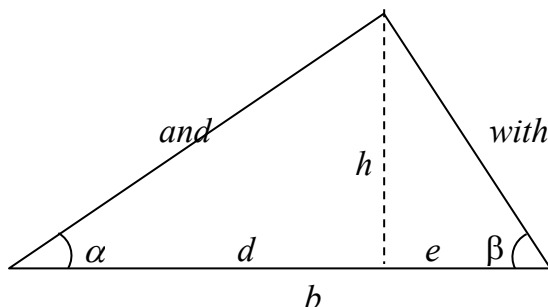


Рис. 89

We need to calculate the length of side c . In the right triangle:

$$c^2 = h^2 + e^2 \quad (\text{by the Pythagorean theorem}).$$

But we do not know the lengths of e and h .

The left triangle will help us calculate h . It is also a right triangle. We know the measures of the angles in this triangle. Therefore, we know the values of their sines and cosines. Consequently,

$$h = a \sin \alpha$$

So, we have calculated the length of the height. But we still need to calculate the length of segment e . Ask yourself, how do you calculate the length of segment e if the lengths of segments d and b are known?

Of course, $e = b - d$.

We'll calculate the length of the side d using the cosine of the known angle in the left triangle. Here,

$$d = a \cos \alpha$$

Therefore,

$$e = b - d = b - a \cos \alpha$$

Now let's substitute the values of the parameters h and e into our first formula ($c^2 = h^2 + e^2$).

$$c^2 = a^2 \sin^2 \alpha + (b - a \cos \alpha)^2$$

Expanding the brackets (the square of the difference)

$$c^2 = a^2 \sin^2 \alpha + b^2 - 2ab \cos \alpha + a^2 \cos^2 \alpha$$

α

We factor out the common factor (a^2) from the first and last terms on the right-hand side of the equation.

$$c^2 = a^2 (\sin^2 \alpha + \cos^2 \alpha) + b^2 - 2ab \cos \alpha$$

Knowing that $\sin^2 \alpha + \cos^2 \alpha = 1$, we get

$$c^2 = a^2 + b^2 - 2ba \cos \alpha$$

We have derived the law of cosines.

It states: *the square of one side of a triangle is equal to the sum of the squares of the other two sides minus twice their product, multiplied by the cosine of the angle between them.*

Now ask yourself: how will the formula change if the angle α is 90° , i.e., for a right triangle?

He's bound to figure it out!

Of course, this will yield the Pythagorean theorem. Consequently, the law of cosines can be called the extended Pythagorean theorem.

So, the length of any side of a triangle is

$$c = \sqrt{a^2 + b^2 - 2ab \cos \alpha}$$

Once again, we have obtained a formula that allows us to calculate what is difficult to measure using what is easy to measure.

Problem.

ΔIn triangle ABC, the length of side AC is 7 cm, and the length of side BC is 4 cm. Angle BAC is 45° . Calculate the length of side AB.



Рис. 90

If the student starts thinking again, remind them that they shouldn't just think. They need to take action!

Let's denote the lengths of the sides and the measures of the angles by parameters.

Remind them that, as a rule, the lengths of the sides are denoted by letters corresponding to the labels of the opposite vertices. Let $AB = c$, $BC = a$, $AC = b$.

And the angles are denoted by Greek letters corresponding to the vertex labels. Let $\angle ABC = \beta$; $\angle BAC = \alpha$; $\angle ACB = \gamma$

We write the formula for the law of cosines:

$$c^2 = a^2 + b^2 - 2ba \cos \alpha$$

Substitute the values of the parameters:

$$c^2 = 4^2 + 6^2 - 2 \cdot 4 \cdot 6 \cdot \cos 60^\circ$$

$$c^2 = 28;$$

$$c = 2\sqrt{7}.$$

The length of side AB is $2\sqrt{7}$ cm.

62. The Sine Theorem

If two angles and the length of one side of a triangle are known, can we calculate the lengths of the remaining two sides? The Law of Sines will help us answer this question.

Let's construct a triangle.

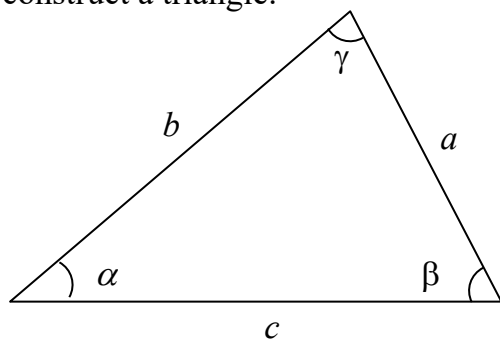


Рис. 91

Let the length of the base be c and the angles adjacent to it be α and β , respectively.

Clearly, the measure of the third angle, γ , can be easily calculated. The sum of the measures of the angles in the triangle, γ , is 180° . Therefore, $\gamma = 180^\circ - (\alpha + \beta)$.

How to calculate the lengths of sides a and b .

Let's draw the altitude. We obtain two right triangles.

In the left triangle, the height is equal to:

$$h = a \sin \beta$$

And in the right one –

$$h = b \sin \alpha$$

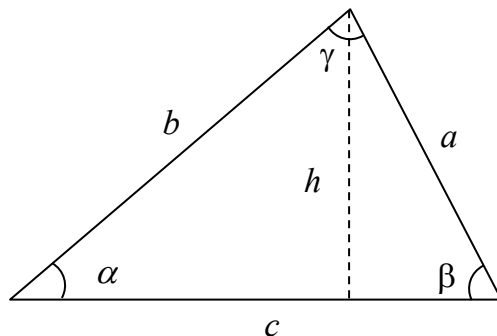


Рис. 92

Since the left-hand sides of the formulas are equal, we can equate their right-hand sides:

$$a \sin \beta = b \sin \alpha$$

Dividing both sides by $(\sin \beta \cdot \sin \alpha)$, we get:

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}$$

Encourage the student to ask: Why did we divide both sides of the equation by $(\sin \beta \cdot \sin \alpha)$?

So far, we've done what's clear and successfully linked the lengths of the two sides and the measures of the two opposite angles. But what about the rest? What should we do with them?

At first, nothing comes to mind. But nothing should come to mind. We just need to do what we know how to do. That is, we continue doing the same thing, only for a different pair of sides and angles.

If nothing helps, ask yourself: is there another way to draw the altitude in this triangle?

Of course you can!

Let's do it.

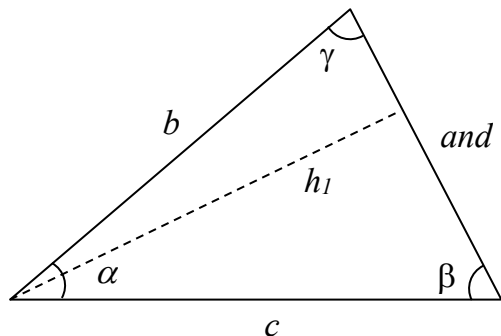


Рис. 93

Let's express the height h_1 in terms of the sines of the corresponding angles β and γ .

$$h_1 = b \sin \gamma$$

$$h_1 = c \sin \beta$$

Since the left-hand sides of the formulas are equal, we can equate their right-hand sides:

$$b \sin \gamma = c \sin \beta$$

Dividing both sides by $(\sin \gamma \cdot \sin \beta)$, we get:

$$\frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

However, above we obtained a different equality of ratios. Let's recall it:

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}$$

Therefore,

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}.$$

This is the - sine theorem. It states: *the ratios of the lengths of the sides and the sines of the opposite angles of a triangle are proportional.*

Therefore, **if we have measured one side of a triangle and two of its angles, then the lengths of the other two sides can be calculated.**

Problem.

In triangle ABC (Δ), the length of side AC is 7 cm. Angle ABC is 30° ($^\circ$), and angle BAC ($^\circ$) is 45° ($^\circ$). Find the length of side BC.

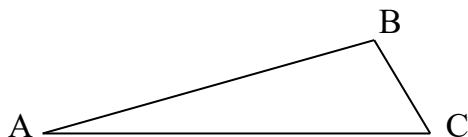


Рис. 94

If our student is lost in thought again, remind them that thinking isn't enough. We need to take action!

In fact, all we need to do each time is translate the problem from the language of words into the language of mathematics. To do this, we simply need to input the parameters and write down the relationships between them (formulas) according to the problem's conditions. Then mathematics should do the rest on its own. That is, we don't solve the problem ourselves (we don't think), but rather use a tool (mathematics) that does all the work for us. A child, on the other hand, tries to think every time—that is, tries to solve the problem on their own, without a tool—and only then writes down the solution using the language of mathematics. It's all backwards.

So, we don't think, we act. Let's denote the lengths of the sides and the measures of the angles by parameters.

Let's remind ourselves once again that, as a rule, the lengths of the sides are denoted by lowercase letters corresponding to the labels of the opposite vertices. Let $AB = c$, $BC = a$, $AC = b$.

Angle measures are denoted by Greek letters. Let $\angle ABC = \beta$; $\angle BAC = \alpha$; $\angle ACB = \gamma$

We write the formula for the law of sines:

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}.$$

We only need the first equality. Let's substitute the values of the parameters into it:

$$\frac{a}{\sin 30^\circ} = \frac{7}{\sin 45^\circ};$$

$$\frac{a}{1/2} = \frac{7}{\sqrt{3}/2};$$

To eliminate the denominator on the left-hand side, we multiply both sides of the equation by 1/2 and obtain

$$a = \frac{7}{\sqrt{3}}.$$

The length of side BC is $\frac{7}{\sqrt{3}}$ cm.

A circle can be drawn through three points that do not lie on a single line. Therefore, a circle can be circumscribed around any triangle. This allows us to generalize the law of sines.

63. The relationship between the central and inscribed angles

We now have enough experience and can try to prove a few more theorems.

Have the child construct a circle and draw a radius, a diameter, and a chord inside it, as shown in the figure below.

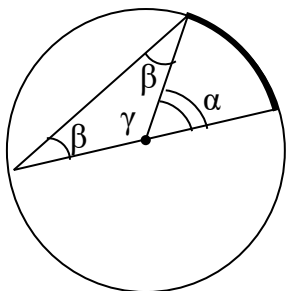


Рис. 95

The sum of the angles in a triangle is 180° , and the sum of the adjacent angles is also 180° . Have the child write the formulas relating the corresponding parameters. This should result in a system of two equations.

$$\begin{aligned} 2\beta + \gamma &= 180^\circ - \text{for the triangle} \\ \alpha + \gamma &= 180^\circ - \text{for adjacent angles} \end{aligned}$$

Now we can eliminate angle γ and derive a relationship between α and β . Have the child subtract the bottom equation from the top one. We get:

$$2\beta - \alpha = 0.$$

Therefore,

$$2\beta = \alpha$$

Now recall that angles formed by two radii are called central angles. In our case, angle α is a central angle.

And what are the angles formed by two chords called? In our case, this is the angle between a chord and a diameter (a diameter is also a chord).

Angles formed by two chords are called *inscribed angles*. In our case, angle β is an inscribed angle.

We have just proven the theorem: *an inscribed angle is equal to half of the central angle subtending the same arc*.

To make sure the child understands everything, ask what the inscribed angle resting on a diameter will be.

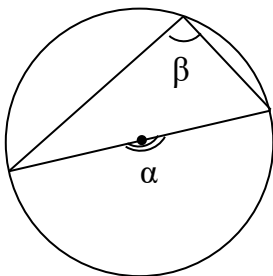


Рис. 96

A diameter consists of two radii, which in turn form a central angle of 180° . The inscribed angle β must be half that. Therefore, it is a right angle! No matter how many inscribed angles resting on a diameter we construct, they will all be right angles.

It follows that, since all inscribed angles subtended by the same arc are equal to half the central angle, they are equal to one another.

Now we can return to the Law of Sines.

Let's draw a circle around triangle ABC and construct the diameter CD, as shown in the figure below. Let's connect the end of the diameter to one of the vertices of the triangle with the chord AD.

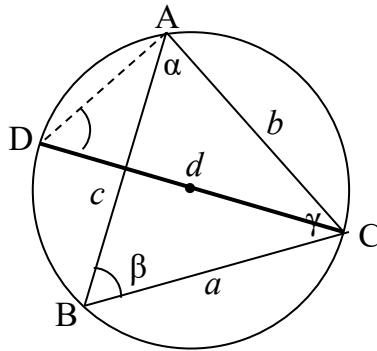


Рис. 97

Angles ADC and ABC are inscribed and rest on the common arc AC. Therefore, they are equal in measure. Thus, their sines are also equal. But angle ADC is inscribed on the diameter, i.e., it is a right angle. Therefore, the sine of angle ADC is equal to the ratio of the opposite side to the hypotenuse, i.e.,

$$\sin \angle ADC = b/d \Rightarrow \sin \beta = b/d \Rightarrow \frac{b}{\sin \beta} = d = 2r.$$

Consequently, we can extend the Law of Sines:

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = 2r.$$

64. Theorem on the angle between a tangent and a chord drawn to the point of tangency

Let's construct a circle. Draw a tangent line (Рис. 98). Have the child describe this line, that is, have them define the term "tangent."

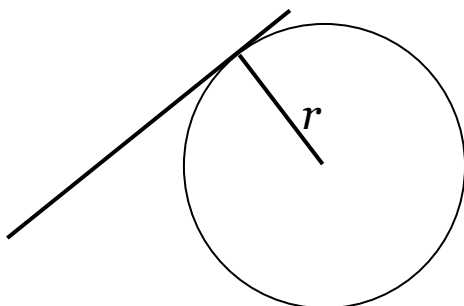


Рис. 98

Make sure they come up with a definition: the shortest possible description.

The result should be as follows.

A tangent is a line that shares one point with the circle.

Now let's draw a radius to the point of tangency.

What can be said about the angle between the radius and the tangent?

Since the tangent and the circle share only one point, the radius drawn to the point of tangency forms two equal adjacent angles with the tangent. Therefore, these angles are straight.

Now let's construct a chord originating from the point of tangency. The chord also forms two adjacent an-

gles with the tangent. Consider the acute angle α (Рис. 99).

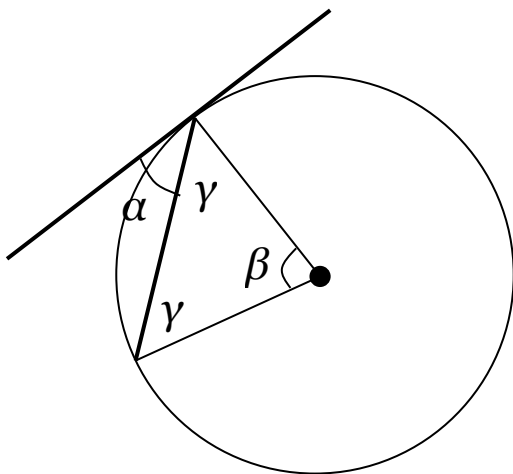


Рис. 99

Here you can ask the child questions to which they already know the answers. Their answers should help them independently derive the theorem on the tangent and the chord.

- Why is the triangle isosceles?
- What is the sum of the angles in a triangle?
- What is the angle between the tangent and the radius?
- What is the sum of angles α and γ ?
- How are angles α and β related?

In fact, since $\alpha + \gamma = 90^\circ$ and $2\gamma + \beta = 180^\circ$, then $\alpha = 2\beta$

The angle between the tangent and the chord drawn from the point of tangency is equal to half the central angle subtended by that chord.

65. Derivation of reduction formulas

But what if the angle is greater than 90° ? For example, what is the sine of 150° ?

But can the sine of for an angle greater than 90° even exist? After all, in this case, the right triangle disappears. And we defined the sine and cosine of an angle based on the fact that the triangle is right-angled. What should we do?

We need to derive a new definition for the sine and cosine of an angle.

Here, the coordinate system will come to our aid.

Let's draw a circle with center at the origin and radius equal to one. The angle α is formed between the radius and the x-axis.

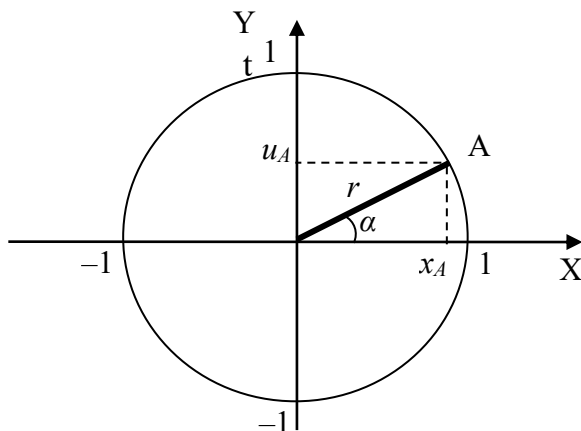


Рис. 100

Let us drop perpendiculars from point A to the x-axis and the y-axis. We obtain two right triangles. The side opposite angle α is equal to the y-coordinate y_A , i.e., the coordinate y_A . And the adjacent side is the x-coordinate x_A .

The sine of angle α is equal to the ratio of the opposite side to the hypotenuse, or the ratio of the y-coordinate of point A to the radius (r), i.e.,

$$\sin\alpha = \frac{y_A}{r} .$$

Accordingly, the cosine of angle α is equal to the ratio of the x-coordinate of point A to the radius (r), i.e.

$$\cos\alpha = \frac{x_A}{r} .$$

In the unit circle, the length of the radius is 1, i.e., $r = 1$.

Therefore,

$$\sin\alpha = y;$$

$$\cos\alpha = x.$$

These conclusions are universal and apply to the calculation of the sine and cosine of any angle.

Now, let's rotate the radius r counterclockwise by another 90 degrees.

We obtain an obtuse angle, equal to $(\alpha + 90^\circ)$.

When rotating the unit radius by an angle of $(\alpha + 90^\circ)$, the new coordinates of its endpoint will be x_B and y_B , respectively.

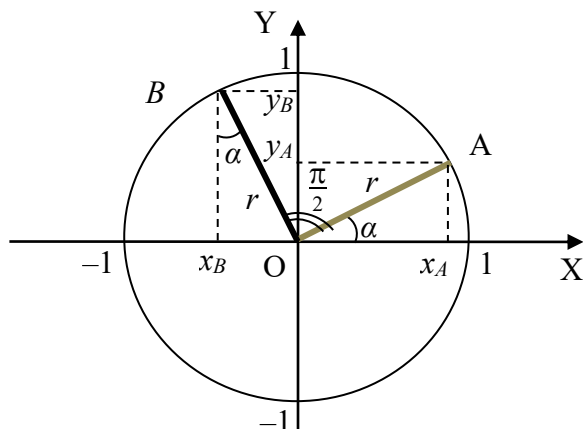


Рис. 101

What next?

Next, we again do only what we can do. We will compare the resulting right triangles. There is nothing else left to do. Remember, we are doing this not because we knew in advance that this is exactly what we needed to do. We simply need to get at least some result to decide how to proceed. Otherwise, we'll be stuck.

There are four right-angled triangles here, and they all have equal hypotenuses and angles. Therefore, they are congruent. The lengths of the legs of these right-angled triangles represent the magnitudes of the corresponding coordinates. And the coordinates are related to each other.

Obviously,

$$x_A = y_B.$$

$$y_A = -x_B \text{ (since the coordinate } x_{(B)} \text{ is negative);}$$

Ask why there is a minus sign in the second equality?

Because the coordinate $x_{(B)}$ is negative.

The student will say that this makes sense.

Check. Have them explain. After all, there is no minus sign in front of the coordinate x_B in the diagram.

Recall that x_B is a parameter of the function, denoting the value of the coordinate. This value, x_B , is negative here. For example, $x_B = -0.7$ (cm). And the value of the coordinate y_A is positive. For example, $y_A = 0.7$ (cm). If we set these coordinates equal, it is clear that $0.7 \neq -0.7$. Therefore, we must place a minus sign in front of one of the parameters so that the equality holds.

Next, let's write down all possible trigonometric formulas for the angle α .

$$y_A = \sin \alpha; \quad -x_B = \sin \alpha;$$

$$x_A = \cos \alpha; \quad y_B = \cos \alpha.$$

Since we said that the cosine of the angle between the unit radius and the x-axis is equal to the x-coordinate of its endpoint, we will assume that

$$\cos(\alpha + 90^\circ) = x_B$$

Recall that this holds only for the unit radius.

Accordingly, we need to make the substitution:

$$\cos(\alpha + 90^\circ) = x_B = -y_A = -\sin \alpha$$

$$\sin(\alpha + 90^\circ) = y_B = y_B = \cos \alpha$$

Finally, we obtain:

$$\cos(\alpha + 90^\circ) = -\sin \alpha$$

$$\sin(\alpha + 90^\circ) = \cos \alpha$$

Let's write down the resulting formulas, replacing degrees with radians.

$$\sin\left(\frac{\pi}{2} + \alpha\right) = \cos \alpha;$$

$$\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin \alpha.$$

These are the so-called reduction formulas.

This refers to reducing trigonometric functions of angles greater than 90 degrees to functions of angles less than 90 degrees.

We have obtained the desired formulas!

On the left-hand side of the formulas are the sine and cosine of an angle greater than 90°, and on the right-hand side are the sine and cosine of an angle less than 90°.

So how do you use them?

It's easy!

For example, let's calculate what the sine of 150° is.

The angle of 150 degrees can be represented as the sum of angles of 90° and 60°. Therefore,

$$\sin 150^\circ = \sin (90^\circ + 60^\circ) = \cos 60^\circ = 0.5.$$

Let's express this in radians

$$\sin \frac{5\pi}{6} = \sin \left(\frac{\pi}{2} + \frac{\pi}{3} \right) = \cos \frac{\pi}{3} = 0.5.$$

Let's note once again that the coordinates of the end of the radius can be either positive or negative. Consequently, the values of the sine and cosine functions can also be either positive or negative. After all, the length of the radius is always positive.

Thus, in the ratios $\frac{y_A}{r}$ and $\frac{x_A}{r}$, the denominator is always positive. The numerator changes sign depending on the value of for angle α .

To remember the sign, the circle is divided into four quadrants, labeled with Roman numerals starting from the

positive value of the OX axis and proceeding counter-clockwise.

First quadrant (I): angles from 0° to 90° .

Here, *the x- and y-coordinates* of the points on the arc of the circle are positive. Consequently, the sines and cosines of the angles between the radii drawn from these points to the x-axis are positive here.

Second quadrant (II): from 90° to 180° .

Here, *the y-coordinates* are positive, and *the x-coordinates* are negative. Consequently, the sine values of angles in the interval from 90° to 180° are positive, and the cosine values are negative.

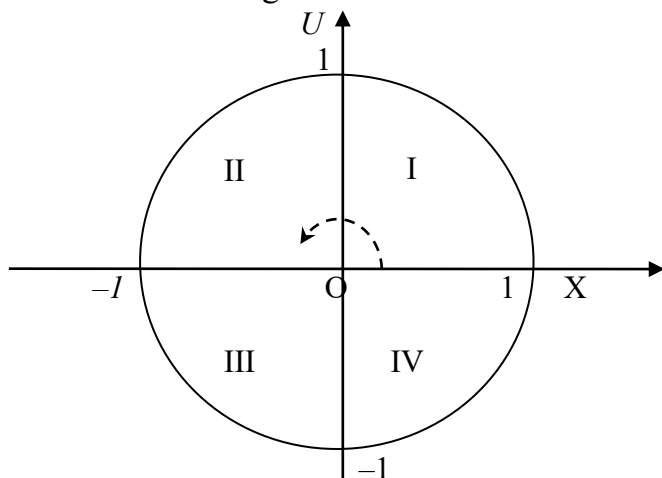


Рис. 102

The third quadrant (III) – from 180° to 270° .

Here, *the y-coordinates* and *x-coordinates* are negative. Consequently, the sine and cosine values of angles in the range from 180° to 270° are negative.

Fourth quadrant (VI) – from 270° to 360° .

Here, *the y-coordinates* are negative, and *the x-coordinates* are positive. Consequently, the sine values of angles in the interval from 270° to 360° are negative, and the cosine values are positive.

What if the angle is greater than 180° ?

For example, what is the sine of for an angle of 210° ?

First, note that this angle falls in the third quadrant. Therefore, both the sine and the cosine of this angle will be negative.

The angle , which is 210° , can be written as the sum $(90^\circ + 90^\circ + 30^\circ)$.

Therefore,

$$\sin 210^\circ = \sin (90^\circ + 90^\circ + 30^\circ).$$

Let's apply the reduction formula

$$\begin{aligned}\sin(90^\circ + (90^\circ + 30^\circ)) &= \cos(90^\circ + 30^\circ) = \\ &= -\sin 30^\circ = -1/2.\end{aligned}$$

Too cumbersome! Let's prove that

$$\begin{aligned}\sin (180^\circ + \alpha) &= -\sin \alpha \\ \cos (180^\circ + \alpha) &= -\cos \alpha\end{aligned}$$

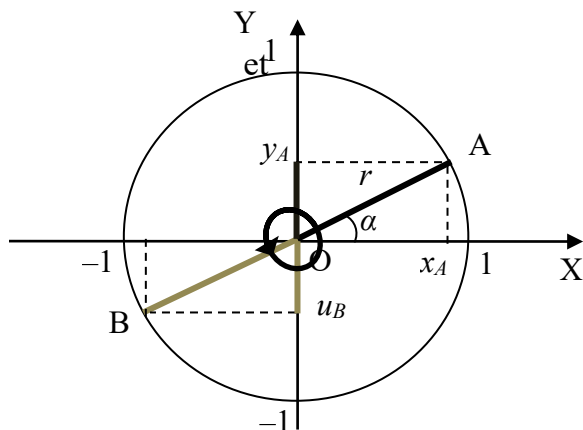


Рис. 103

Rotate the radius OA by 180° . The new radius OB forms an angle $(\alpha + 180^\circ)$ with the axis OX . The sine of this angle is equal to the projection of $O_{u(B)}$. The projection is negative and, in absolute value, equal to the projection of $O_{u(A)}$, which in turn is equal to the sine of the angle α . Therefore,

$$\sin(180^\circ + \alpha) = -\sin \alpha$$

$$\cos(\pi + \alpha) = -\cos \alpha .$$

For angles greater than 360° , we simply subtract the period, for example,

$$\sin(360^\circ + \alpha) = \sin \alpha$$

$$\cos(360^\circ + \alpha) = \cos \alpha$$

Let's examine this in more detail and figure out what a period is.

66. Measuring angles greater than 360 degrees

Construct the angle and ask to measure it. With a protractor at hand, this is not difficult to do. But will you be able to correctly state or write down the result of this measurement? The question may seem a little strange at first. Nevertheless, the student will confidently answer.

– Of course, I can. Here, I've placed the protractor and the angle has been measured. It is thirty degrees.

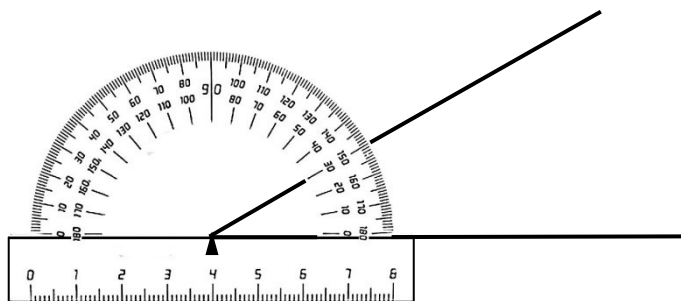


Рис. 104

– Yes, but that's just the acute angle .
– Oh, right, there's also the obtuse angle . It's 150° .
– Really? Move the protractor . See? There is no obtuse angle equal to 150° on the drawing.

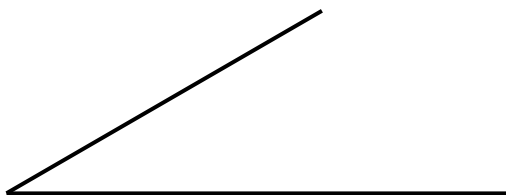


Рис. 105

Let's replace the protractor .

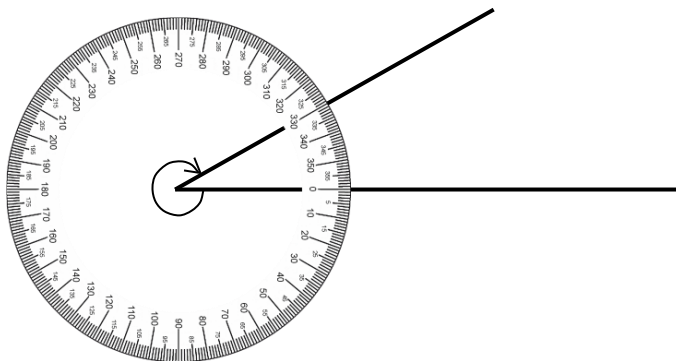


Рис. 106

— Really! Here's a 30-degree angle° and a angle of... 330° .

— That's right. Is that all? Are there no more angles?

— I don't think so!

— That's just it—it seems that way. Look more closely. There are a lot of them there.

At first, children learn to distinguish only acute angles. Then they begin to see obtuse angles, i.e., angles greater than 90° , but less than 180° . They just noticed that there are angles greater than 180° . Now we need to show that there are angles greater than 360° and less than 0° .

Let's look at a clock face.

We know that a day begins at 0 o'clock. At this moment, both hands—the minute hand and the hour hand—point to 12.

How many degrees will the minute hand rotate in one hour?

Exactly 360 degrees, i.e., it will make one full revolution. The number of degrees in one revolution is called a period.

And in 2 hours? Clearly, two times 360 degrees. That's a total of 720 degrees. That's already two full revolutions (periods).

And if 3 hours pass? Then, 1080 degrees. That is three periods.

When the minute hand makes one revolution, the hour hand rotates by one twelfth of a circle, i.e., it shows that exactly 1 hour has passed. Therefore, if the hour hand points to 7 o'clock, then the minute hand has made seven revolutions, i.e., it has rotated by $7 \times 360^\circ = 2520^\circ$.

Let's remove the hour hand. We'll leave only the minute hand. How many degrees is it currently rotated from zero?



Рис. 107

It is impossible to say for certain. This is because it could be 0° , 360° , 2520° , or any other angle that is a multiple of 360° . The set of possible values for this angle consists of $360^\circ \cdot n$, where n is an integer.

In radians, this angle is equal to $2\pi n$, where n is an integer .

Let's consider the unit circle once more.

Let's construct the radius . Let the angle between the radius and the x-axis be α . What could this angle be if measuring it with a protractor gave 30° ?

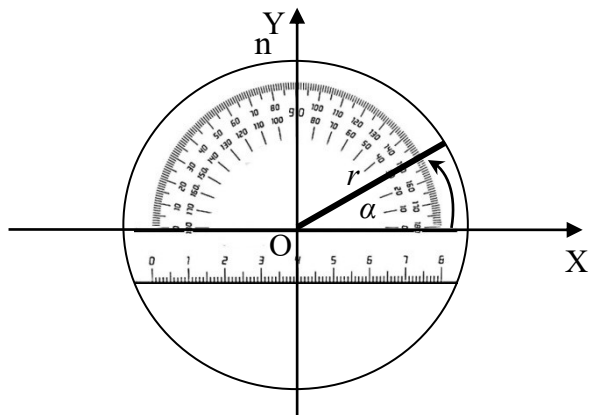


Рис. 108

It is clear that $\angle \alpha = 30^\circ$.

It is also true that $\angle \alpha = 390^\circ$, (i.e., $30^\circ + 360^\circ$). It may also turn out that $\angle \alpha = 750^\circ$ and so on. The entire set of possible values for this angle can be written as follows:

$\angle \alpha = 30^\circ + 360^\circ \cdot n$, where n is an integer.

Here, 360° is the period. And n is the number of periods.

In radians, the result will be:

$$\angle \alpha = \frac{\pi}{6} + 2\pi n, \text{ where } n \text{ is an integer.}$$

In radians, the period is equal to $2\pi \text{ rad.}$ (i.e., 6.28 rad.). Note that this is if the radius rotates counterclockwise, i.e., in the positive direction.

What would the angle between the radius and the x-axis be if the radius rotated clockwise?

67. Negative angles

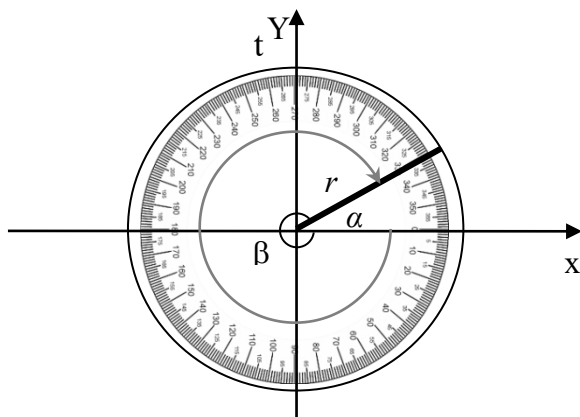


Рис. 109

The angle β , measured in the opposite direction (clockwise), will be denoted by a minus sign, i.e., it will be negative.

In our case, let us denote the angle by the parameter β .

Clearly, $\angle\beta = -330^\circ$. But, similarly, it is true that $\angle\beta = -690^\circ$, (i.e., $-330^\circ + (-360^\circ)$), and so on.

The entire set of possible values for the angle can be written as follows:

$$\angle\beta = -330^\circ - 360^\circ \cdot n, \text{ where } n \text{ is an integer.}$$

Let's express the angle β in terms of the angle α :

$$\angle\beta = \alpha - 360^\circ \cdot n$$

Is it possible to write all possible values of the angles in a single line?

Of course, we can.

Let α be the parameter denoting the angle between the x-axis and the radius, measured with a protractor. Then all possible values of the angles can be written as:

$$\alpha + 2\pi n, \text{ where } n \text{ is any integer.}$$

But we had exactly the same expression for positive angles.

That's absolutely correct. But in the previous case, we said that n is an integer. Students usually understand "integers" to mean only positive numbers. Now, let's note that n is any integer, meaning positive, negative, and zero.

To denote all integers (positive, negative, and zero), the parameter Z is used.

Consequently, the final notation for all possible values of the angle between the radius and the OX axis in this figure will be:

$$\alpha + 2\pi n, (n \in Z).$$

Let's review this once more. On the unit circle, positive angles are measured counterclockwise: $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}, 2\pi, \dots$

Negative angles are measured clockwise: $0, -\frac{\pi}{2}, -\pi, -\frac{3\pi}{2}, -2\pi, -\frac{5\pi}{2}, -2\pi, \dots$

68. Periods of the sine and cosine of an angle

Let's take another look at the unit circle.

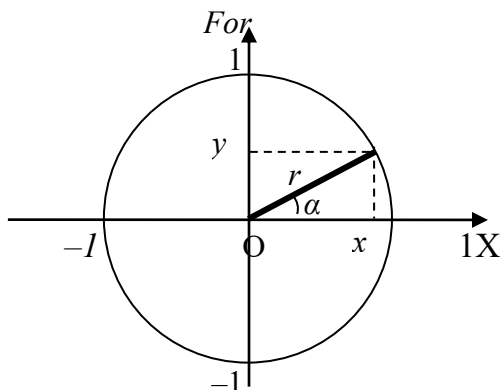


Рис. 110

Here, the angle between the radius r and the x -axis is equal to

$$\alpha + 2\pi n, (n \in \mathbb{Z}).$$

We know that the sine of this angle, for any values of $n \in \mathbb{Z}$, is equal to the ordinate “ y ” of the endpoint of the radius, and the cosine is equal to the abscissa “ x ”.

Therefore,

$$\sin(\alpha + 2\pi n) = \sin \alpha; \quad \cos(\alpha + 2\pi n) = \sin \alpha$$

This can be seen in the diagram.

Since $2n$ is an even number for any $n \in \mathbb{Z}$, we can say that this equality holds for any even number π .

For example,

$$\sin(\alpha + 8\pi) = \sin \alpha; \quad \cos(\alpha + 8\pi) = \sin \alpha$$

And for an odd number?

Since $(2n+1)$ is an odd number for any $n \in \mathbb{Z}$, then

$$\sin(\alpha + (2n+1)\pi) = -\sin \alpha$$

$$\cos(\alpha + (2n+1)\pi) = -\cos \alpha$$

Let's illustrate this on the unit circle.

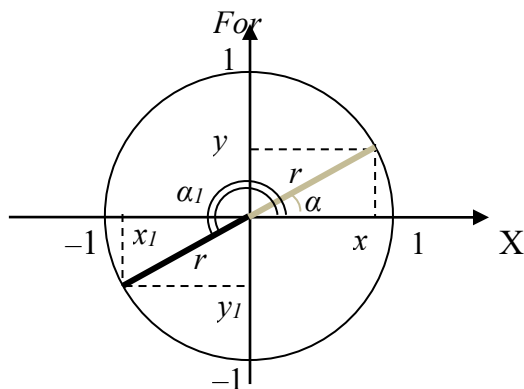


Рис. 111

Here, $\alpha_1 = \alpha + \pi$.

Or, for all values of the angles between the given radii and the OX axis, we can write:

$$\alpha_1 = (\alpha + 2\pi n) + \pi = (\alpha + (2n+1)\pi).$$

Therefore,

$$\sin \alpha_1 = \sin(\alpha + (2n+1)\pi).$$

However, the diagram shows that the sine of the angle α_1 is equal to the y-coordinate y_1 . And the y-coordinate y_1 , in turn, is equal to the y-coordinate with the opposite sign. And the y-coordinate is equal to the sine of the angle α . Therefore,

$$\sin \alpha_1 = \sin(\alpha + (2n+1)\pi) = y_1 = -y = -\sin \alpha.$$

Or, for an odd number π

$$\sin (\alpha + (2n+1)\pi) = -\sin \alpha .$$

Similarly, for the cosine of the angle α_1 .

$$\cos (\alpha + (2n+1)\pi) = -\cos \alpha .$$

For example,

$$\sin (\alpha + 5\pi) = -\sin \alpha ;$$

$$\cos (\alpha + 5\pi) = -\cos \alpha .$$

In fact, the angle $(\alpha + 2\pi n)$ lies in the first quadrant, while the angle $\alpha + (2n+1)\pi$ lies in the third quadrant. And we know that in the first quadrant, the sine and cosine values of angles are positive, while in the third quadrant, they are negative.

69. How do you solve trigonometric equations?

Is it possible to calculate the measure of an angle if the sine of that angle is known?

Most likely, yes.

$$\text{Let } \sin \alpha = \frac{1}{2}.$$

What will the angle α be equal to?

Clearly, $\angle \alpha = 30^\circ$.

Are there any other possible values?

We already know that there are. We need to add turns, i.e., periods. Therefore, for our case

$$\angle \alpha = 30^\circ + 2\pi n, \quad (n \in \mathbb{Z})$$

It seems to be correct. But let's check the result using the unit circle. The sine of the angle of inclination of the unit radius to the OX axis on the unit circle is equal to the projection of this radius onto the OY axis. In our case, this projection is equal to one-half.

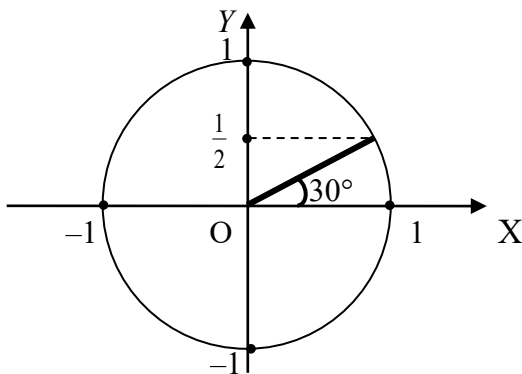


Рис. 112

However, another radius, , also has a projection equal to one-half. This radius forms an obtuse angle with the OX axis.

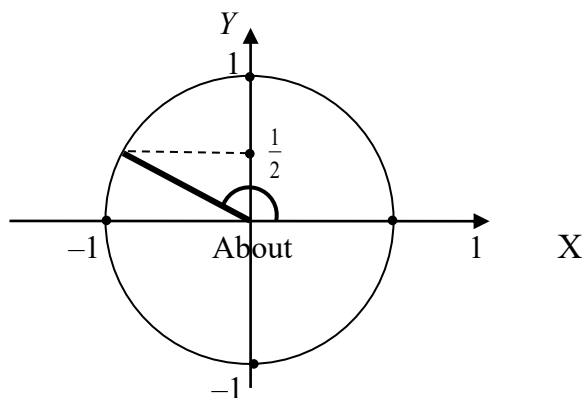


Рис. 113

What should I do?

We need to find the second root of the equation.

But how do we write it down?

Let's examine both angles on the same diagram. Here, the dotted line passing through the sine value of the angle and parallel to the x-axis is called the sine line.

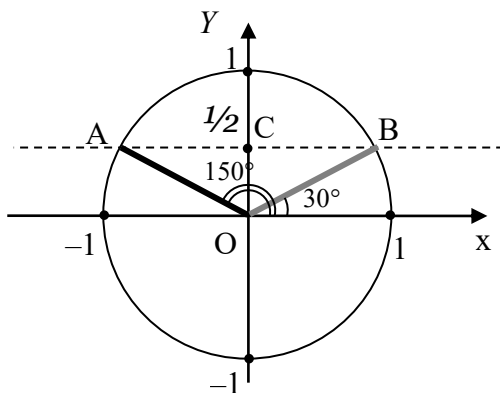


Рис. 114

The angle between the radii is 120° . Clearly, the angle between the x-axis and the radius OA is equal to $120^\circ + 30^\circ = 150^\circ$. Or $180^\circ - 30^\circ = 150^\circ$.

Therefore, the angle, whose sine is equal to one-half, is the angle at 150° . This is the second root of the equation.

Thus, all solutions to our equation can be written as:

$$\angle \alpha = 30^\circ + 2\pi n \text{ and } \angle \alpha = 150^\circ + 2\pi n.$$

And how do we write the solution in general form?

Let $\sin \alpha = t$. The roots of this equation are the following angles:

$$\alpha = \arcsin t + 2\pi n$$

and

$$\alpha = (\pi - \arcsin t) + 2\pi n$$

The arcsine ($\arcsin$) of a number (t) is defined as the angle, whose sine is equal to that number (t).

Is it possible to write the solution to this equation in a single line?

Of course, it can.

$$\alpha = (-1)^n \arcsin t + \pi n, \quad (n \in \mathbb{Z}).$$

What does this mean?

Let's apply this formula to our specific case.

$$\sin \alpha = \frac{1}{2}.$$

$$\alpha = (-1)^n \arcsin \frac{1}{2} + \pi n, \quad (n \in \mathbb{Z}).$$

or

$$\alpha = (-1)^n \frac{\pi}{6} + \pi n, \quad (n \in \mathbb{Z}).$$

$$\text{If } n=0, \text{ then } \alpha = (-1)^0 \frac{\pi}{6} + \pi \cdot 0 = \frac{\pi}{6};$$

$$\text{or in degrees, } \alpha = (-1)^0 30^\circ + 180^\circ \cdot 0 = 30^\circ.$$

$$\text{If } n=1, \text{ then } \alpha = (-1)^1 \frac{\pi}{6} + \pi \cdot 1 = -\frac{\pi}{6} + \pi = \frac{5\pi}{6};$$

$$\text{or } \alpha = (-1)^1 30^\circ + 180^\circ \cdot 1 = -30^\circ + 180^\circ = 150^\circ.$$

$$\text{If } n=2, \text{ then } \alpha = (-1)^2 \frac{\pi}{6} + \pi \cdot 2 = \frac{\pi}{6} + \pi \cdot 2 = \frac{13\pi}{6};$$

$$\text{or } \alpha = (-1)^2 30^\circ + 180^\circ \cdot 2 = 30^\circ + 360^\circ = 390^\circ,$$

and so on.

Now, let our student try performing the same operations for negative values of the variable n .

And how do we calculate the measure of an angle if the cosine of the angle is known?

Let,

$$\cos \alpha = \frac{1}{2}$$

We know that the cosine of 60° is equal to one-half. Therefore, we already have the first root of .

$$\alpha = 60^\circ + 2\pi n, \quad (n \in \mathbb{Z}).$$

To calculate the second root of , let's consider the unit circle.

The cosine of an angle is equal to the projection of the unit radius onto the x-axis. Since the cosine of our an-

gle is equal to one-half, we will draw the cosine line through the coordinate $x = \frac{1}{2}$

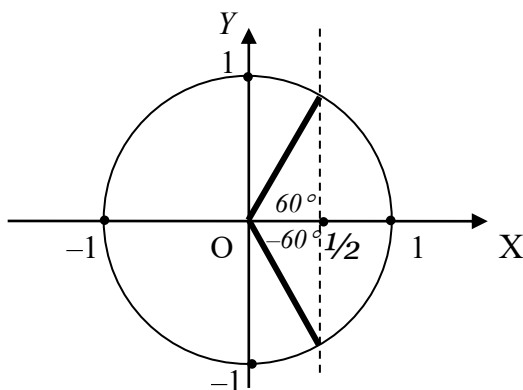


Рис. 115

The diagram shows that the cosines of two angles, 60° and -60° , have the value $\frac{1}{2}$.

Therefore, the second root of is:

$$\alpha = -60^\circ + 2\pi n, (n \in \mathbb{Z}).$$

Let's write the solution to the equation in a single line:

$$\alpha = \pm 60^\circ + 2\pi n, (n \in \mathbb{Z}).$$

In general, the solution to the equation $\cos \alpha = t$ will be:

$$\alpha = \pm \arccos t + 2\pi n, (n \in \mathbb{Z}).$$

The arccosine ($\arccos$) of a number (t) is defined as the measure of the angle, whose cosine is equal to that number (t).

70. Solving trigonometric equations within a given angle interval

Suppose there are additional conditions for solving trigonometric equations. For example, suppose we need to select from the solution only those angles that lie within a given interval.

For example, we need to solve the equation

$$\sin \beta = \frac{1}{2}$$

Additionally, from the entire set of solutions, we need to select angles located in the interval $[\frac{\pi}{2}; \frac{3\pi}{2}]$

Let's write down the solution to the equation.

$$\beta_1 = \arcsin 0.5 + 2\pi n = \frac{\pi}{6} + 2\pi n, (n \in \mathbb{Z})$$

$$\beta_2 = \pi - \arcsin 0.5 + 2\pi n = \frac{5\pi}{6} + 2\pi n, (n \in \mathbb{Z})$$

Let's construct a unit circle.

Draw the sine line through the ordinate equal to $\frac{1}{2}$.

Draw radii to the points of intersection of the sine line and the circle. The sines of the angles between them and the OX axis will be equal to $\frac{1}{2}$.

Let's mark the interval $[\frac{\pi}{2}; \frac{3\pi}{2}]$ with a dotted line, from the smaller angle to the larger one, counterclockwise.

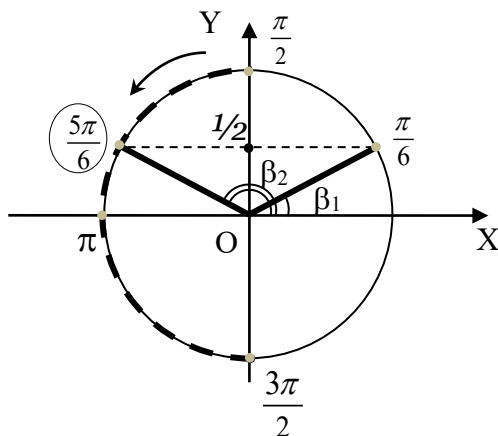


Рис. 116

The angles $\frac{\pi}{6}$, and all angles equal to $\frac{\pi}{6} + 2\pi n$, and n do not fall within this interval. However, the angle $\frac{5\pi}{6}$ does fall within it.

Therefore, the solution to the equation is the set of angles

$$\beta_1 = \frac{\pi}{6} + 2\pi n, \quad \beta_2 = \frac{5\pi}{6} + 2\pi n, \quad (n \in \mathbb{Z})$$

or in a single line

$$\beta = (-1)^n \frac{\pi}{6} + \pi n, \quad (n \in \mathbb{Z})$$

And within the given interval, only one angle from this set of angle values falls within it :

$$\beta = \frac{5\pi}{6}.$$

Let's illustrate this solution on a graph.

Let's plot the graph of for the function $y = \sin \beta$

Let's plot the angles, expressed in radians, on the x-axis, and the sine values of these angles on the y-axis.

Since in our equation $y = \sin(\beta) = 0.5$, we will draw a line (dotted) parallel to the x-axis through the ordinate equal to $\frac{1}{2}$.

The solutions to our equation are the coordinates of the projections of the points of intersection of this line and the graph of $y = \sin \beta$. In our case, there is only one such solution: $\beta = \frac{5\pi}{6}$, since only this angle falls within the given interval $[\frac{\pi}{2}; \frac{3\pi}{2}]$.

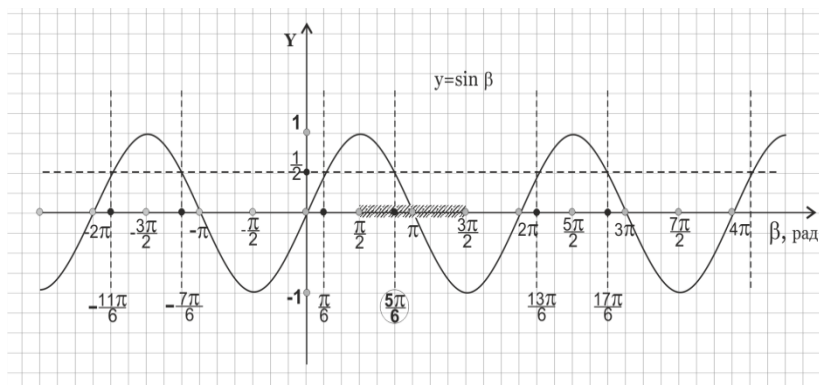


Рис. 117

Let's solve one more equation

$$\cos \gamma = -\frac{\sqrt{3}}{2}.$$

Let's write down the solution to the equation.

$$\gamma = \pm \arccos\left(-\frac{\sqrt{3}}{2}\right) + 2\pi n = \pm \frac{5\pi}{6} + 2\pi n, (n \in \mathbb{Z})$$

Suppose that, from the entire set of these solutions, we need to select angles located in the interval $[-\pi; \frac{\pi}{2}]$.

Let us construct a unit circle.

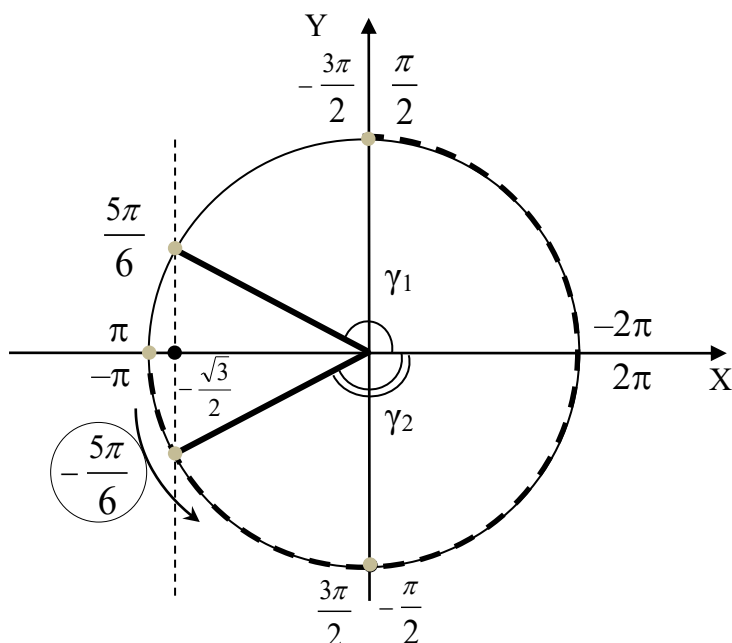


Рис. 118

Draw the cosine line through the point with an abscissa of $\left(-\frac{\sqrt{3}}{2}\right)$. Draw radii to the points of intersection of the cosine line and the circle. The cosines of the angles between these points and the x-axis will be $\left(-\frac{\sqrt{3}}{2}\right)$.

Let's mark the interval $\left[-\pi; \frac{\pi}{2}\right]$, from the smaller angle to the larger one, counterclockwise.

Only the angle $\gamma_1 = \frac{2\pi}{3}$ falls within this interval.

Let's plot this solution.

Let's plot the graph of the function $y = \cos \gamma$.

Let's plot the angles, expressed in radians, on the x-axis, and the cosines of these angles on the y-axis.

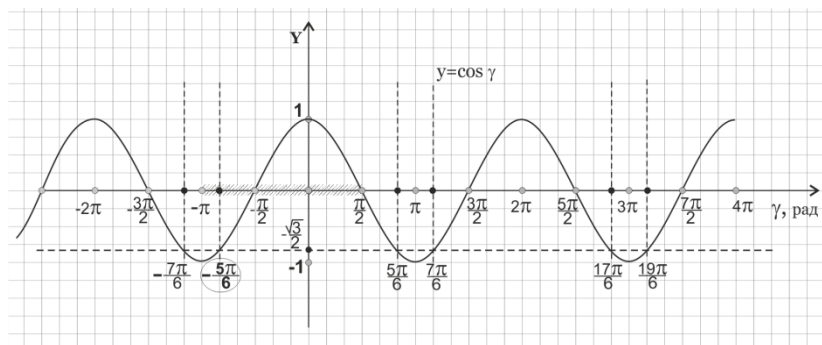


Рис. 119

Since in our equation $y = \cos \gamma = -\frac{\sqrt{3}}{2}$, we will draw a line (dotted) parallel to the y-axis through the x-coordinate equal to $\left(-\frac{\sqrt{3}}{2}\right)$.

The solutions to our equation are the coordinates of the projections of the points of intersection of this line and the graph of $y = \cos \gamma$.

These are the angles $\gamma = \pm \frac{5\pi}{6} + 2\pi n, (n \in \mathbb{Z})$

Only one angle $\gamma = -\frac{5\pi}{6}$ falls within the given interval $[-\pi; \frac{\pi}{2}]$.

71. What is the derivative of a function

Formulas and functions are becoming increasingly complex, while the requirements for measurement and calculation accuracy are rising. At some point, the mathematical tools we know may prove too crude, and we will need a more precise and powerful tool.

Consider the motion of a car. Its coordinates change over time. The change in the coordinate per unit of time is called the rate of change of the coordinate. We know this.

In uniform linear motion, the change in the coordinate is equal to the displacement. We know this, too. Here, the displacement per unit of time is called the speed of motion. In our case, the motion is uniform, so the speed is a constant. This means that the speed is the same at every point. That is, calculating the speed at time t is straightforward.

But what if the velocity changes? How do we calculate the value of the velocity at time t in this case?

The change in the coordinate at a point is zero. Therefore, the velocity at that point is also zero. The same is true at the neighboring point. At all points, the velocity is zero. This follows from the formula $v = \Delta x / \Delta t$. Consequently, at every point, the displacement is zero ($s = \Delta x = 0$). So how does the car move? And how do we calculate the velocity at a given point? It turns out that the old method—dividing displacement by time—is not suitable for calculating velocity here.

Mathematicians have found a solution. It is roughly the same as in the case of deriving the formula for the area of a circle (рис. 40).

We know that displacement is a function of time, i.e., $s = f(t)$. Consequently, the velocity at a given point is the ratio of displacement to the time interval, which tends to zero. Only in this case do we obtain the velocity at a given point, since the displacement becomes infinitely small, i.e., reduces to a point.

In parametric form, this is written as:

$$v(x_0) = \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t}$$

This reads as follows: the velocity (v) at a given point with coordinates x_0 is equal to the limit of the ratio of the change in displacement to an infinitesimally small time interval. This is the limit ratio or **derivative**.

Calculating the derivative is a mathematical operation. The derivative is the rate of change of the function at a given point. It is denoted by a prime symbol to the upper right of the function. For example, $y' = f'(x)$ means that we need to take the derivative of the function $y = f(x)$.

So how do you take it?

Let's consider the general case. Let there be a function $y = f(x)$. Its rate of change at the point x_0 will be equal to:

$$y' = f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta f(x)}{\Delta x}$$

This is the derivative. *The derivative of a function at the point x_0 is defined as the rate of change of the function at that point, and it is equal to the limit of the ratio of the change in the function to the change in the argument as the change in the argument approaches zero.*

How is it calculated?

We know that the change in a quantity is the difference between its final and initial values. Therefore, $\Delta f(x) = f(x) - f(x_0)$ and $\Delta x = x - x_0$.

Since $x = x_0 + \Delta x$, we can write

$$\Delta f(x) = f(x) - f(x_0) = f(x_0 + \Delta x) - f(x_0).$$

Substitute this value into the formula for the derivative.

$$\begin{aligned} f'(x_0) &= \lim_{\Delta x \rightarrow 0} \frac{\Delta f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x) - f(x_0)}{\Delta x} = \\ &= \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \end{aligned}$$

So what do we do about this? For example, how do we find the derivative of the function $f(x) = x^2$ at the point $x = 0$?

Let's substitute the value of the function itself at the initial point (x_0) and the final point ($x = x_0 + \Delta x$) into the formula for calculating the derivative of f .

$$\begin{aligned} f'(x_0) = (x^2)' &= \lim_{\Delta x \rightarrow 0} \frac{(x_0 + \Delta x)^2 - x_0^2}{\Delta x} = \\ &= \lim_{\Delta x \rightarrow 0} \frac{x_0^2 + 2x_0\Delta x + \Delta x^2 - x_0^2}{\Delta x} = \\ &= \lim_{\Delta x \rightarrow 0} \frac{2x_0\Delta x + \Delta x^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2x_0\Delta x}{\Delta x} + \lim_{\Delta x \rightarrow 0} \frac{\Delta x^2}{\Delta x} = \\ &= \lim_{\Delta x \rightarrow 0} 2x_0 + \lim_{\Delta x \rightarrow 0} \Delta x = \lim_{\Delta x \rightarrow 0} 2x_0 = 2x_0. \end{aligned}$$

We have calculated the derivative of the function $f(x) = x^2$ at the point x_0 . It is equal to $2x_0$. In exactly the same way, we can calculate the derivative of the function $f(x) = x^3$. It will be equal to $3x^2$.

In general form:

$$(x^n)' = nx^{n-1}$$

For example, let's calculate the velocity of a body 3 seconds after it was thrown vertically upward with an initial velocity of equal to 40 m/s?

From the physics course, the formula for calculating the position of a uniformly accelerated body looks like this:

$$x = x_0 + v_0t + at^2/2.$$

Here, the coordinate x is a function, and time t is the argument. Let's calculate the derivative of this function, substitute our data into it, and obtain the velocity of the body at time t .

$$v = x' = (x_0 + v_0t + at^2/2)' = v_0 + at.$$

Here, $v_0 = 40$ m/s; $a = -9.8$ m/s² (acceleration due to gravity); $t = 3$ s.

Therefore,

$$v = 40 \text{ m/s} - 9.8 \text{ m/s}^2 \cdot 3 \text{ s} = 10.6 \text{ m/s}.$$

Is there another way to calculate the derivative? Yes, there is. Geometrically.

Let's consider a geometric representation of the derivative. In рис. 120, the graph of the function $y=f(x)$ is shown. A tangent line is drawn to the graph at point A.

Let the derivative of this function at the point with coordinates x_0 be

$$y' = f'(x_0).$$

The abscissa x_0 corresponds to the ordinate y_0 .

Let us take a point on the x-axis with coordinates x such that $x > x_0$. The x-coordinate x corresponds to the y-coordinate y .

Here,

$$\Delta x = x - x_0$$

$$\Delta y = y - y_0$$

The derivative of the function $y' = f'(x_0)$ is, by definition, equal to:

$$y' = f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

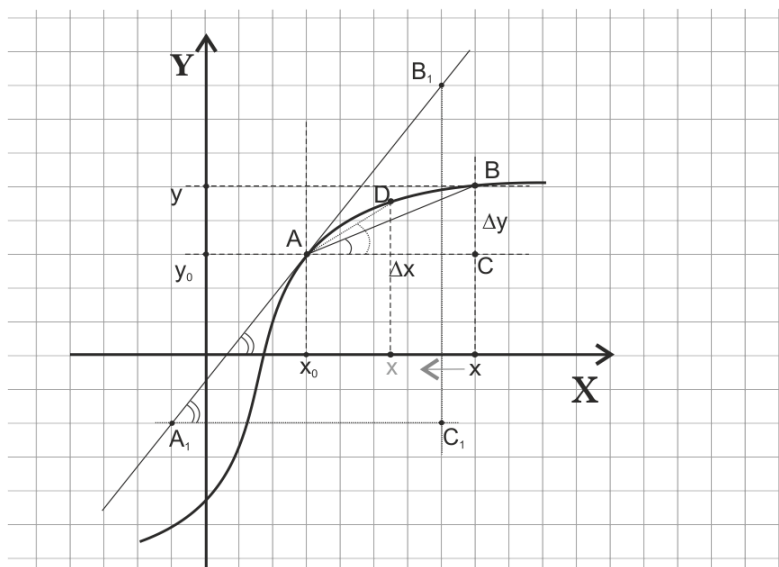


Рис. 120

So far, everything is clear. The only thing that is unclear is why the tangent line is here. Let's not rush. Let's continue.

Let's connect points $A(x_0; y_0)$ and $B(x; y)$ with a line segment. In right triangle ABC

$$\frac{\Delta y}{\Delta x} = \operatorname{tg}(\angle BAC)$$

Since $\Delta x \rightarrow 0$, we have $\angle BAC \rightarrow \angle B_1 A_1 C_1$
Why?

For example, if we halve Δx , i.e., move the point x to the left by half the length of the segment to x_0 , we obtain the angle $\angle DAC$. It is obviously larger than the angle $\angle BAC$. As Δx continues to decrease, the angle will continue to grow and, in the limit, when $\Delta x \rightarrow 0$, tends toward the angle $\angle B_1 A_1 C_1$. Therefore,

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \tan(\angle B_1 A_1 C_1)$$

This is the derivative of the function at the point x_0 .

Let's compare the angle $\angle B_1 A_1 C_1$ and the angle of inclination of the tangent at the point x_0 to the x -axis. They are equal.

Therefore, *the derivative of the function at the point x_0 is equal to the tangent of the angle of inclination of the tangent to the graph of this function at the point x_0 .*

The tangent of the angle of inclination of the tangent can be easily calculated from the graph. We need to find a right triangle with a hypotenuse lying on the tangent such that its vertices have integer coordinates. In our diagram, there are two such triangles. These are $\triangle A_1 B_1 C_1$ and $\triangle AB_1 C$.

$$\tan(\angle B_1 A_1 C_1) = \tan(\angle B_1 A C) = \frac{5}{4} = 1.25.$$

For each of these triangles, the angle between the hypotenuse and the base of the triangle is equal to the angle of inclination of the tangent.

Consequently, the derivative of our function at the point x_0 is also equal to 1.25.

Thus,

$$y'=f'(x_0)=\lim_{\Delta x\rightarrow 0}\frac{\Delta y}{\Delta x}=\operatorname{tg}\alpha$$

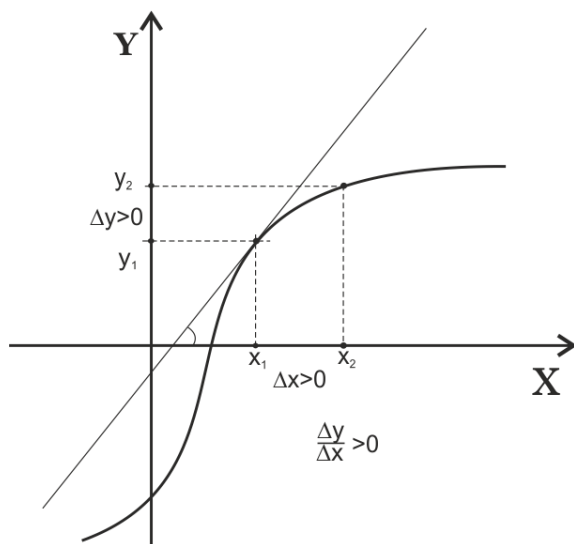
72. Analyzing a function by calculating its derivative

If we have a graph of the function , analyzing it is straightforward. But what if there is no graph?

In fact, it may happen that plotting the graph of the function is quite difficult, but some of its properties need to be calculated quickly. Here, the derivative of the function will help us.

For example, how to determine the intervals where the function is increasing and decreasing. Let's examine how the derivative of the function behaves on these intervals.

1. Let the function be increasing on some interval. Let's take any two points x_1 and x_2 on the x-axis within this interval such that $x_2 > x_1$



Here, $x_2 - x_1 = \Delta x > 0$, and $y_2 - y_1 = \Delta y > 0$.
Therefore,

$$y' = f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} > 0.$$

That is, the derivative of an increasing function is always positive.

The angle of the slope of the tangent line to any point on the graph of this function in the given interval is acute, i.e., lies in the range from 0° to 90° . The tangent of this angle is also positive.

So, if the derivative of the function is positive on some interval, then the function is increasing on that interval.

So what do we do with this?

This rule allows us to determine the intervals where the function is increasing without plotting its graph . First, we need to find the derivative of the function. Then, we need to find the intervals of argument values for which this derivative is greater than zero, i.e., solve the inequality

$$y' = f'(x_0) > \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} > 0.$$

2. Let the function be decreasing on some interval (Fig. 122). Let us take any two points x_1 and x_2 on the x-axis within this interval such that $x_2 > x_1$.

Here, $x_2 - x_1 = \Delta x > 0$, but $y_2 - y_1 = \Delta y < 0$

Therefore,

$$y' = f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} < 0.$$

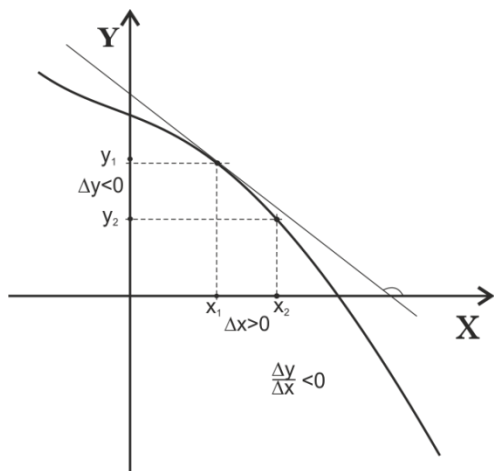


Рис. 122

The derivative of a decreasing function is always negative.

The angle of the slope of the tangent line to any point on the graph of this function in a given interval is obtuse, i.e., lies in the range from 90° to 180° . The tangent of this angle is also negative.

Consequently, if the derivative of the function is negative on a certain interval, then the function is decreasing on that interval.

To determine the intervals where the function is decreasing without plotting its graph, one must first calculate the derivative of the function. Then, determine the intervals of argument values for which this derivative is less than zero, i.e., satisfy the inequality

$$y' = f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} < 0.$$

3. What is the derivative of at the points where the function changes from increasing to decreasing, and vice

versa? Here, , the derivative is equal to zero. The values of the function at these points are called **maxima** or **minima** of the function. The points themselves (the x-coordinates) are called **points of maximum** and **points of minimum**.

The tangents to the graph of the function at its maxima or minima are parallel to the OX axis, i.e., their angles of inclination to the OX axis are zero. Consequently, the tangents of the angles of inclination are also zero.

Let the function $y = f(x)$ be continuous in the neighborhood of the point x_0 . If, as the the argument passes through this point, the sign of the derivative changes from positive to negative, then this is a local maximum.

If the sign of the derivative of the function changes from negative to positive, then this is a local minimum.

To find *the minimum of the function* , you must find *the minimum point* and substitute its value into the equation of the function. The calculated value of the function at the minimum point will be *the minimum of the function*.

To find *the maximum of the function* ` ` , you must find *the maximum point* and substitute its value into the equation ` ` . The calculated value of the function at the maximum point is *the maximum of the function*.

Therefore, one must distinguish between the *maximum point* and *the maximum of the function*.

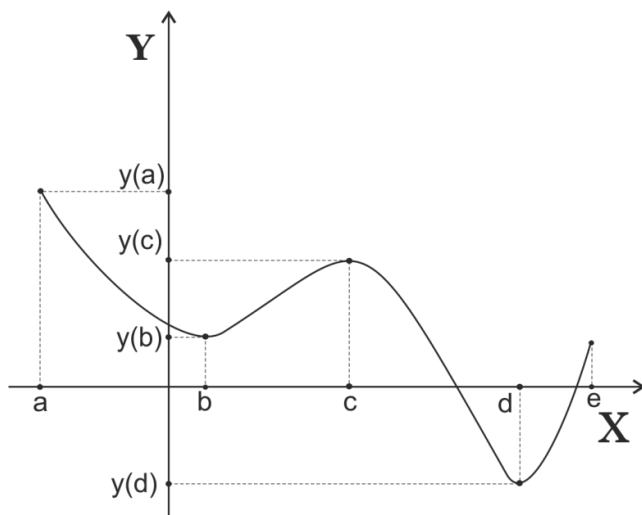


Рис. 123

At REF _Ref409885609 * Lower \h \r * MERGEFORMAT (Fig. 123), the graph of a certain function defined on the interval $[a; e]$ is shown.

Here,

the abscissa b is a local minimum point ;

the ordinate $y(b)$ is the local minimum of the function ;

the x-coordinate of c – the point of local maximum ;

ordinate $y(c)$ – local maximum of the function ;

x-coordinate d – the point of minimum of the function ;

y-coordinate $y(d)$ – the minimum of the function

y-coordinate $y(a)$ – the maximum value of the function.

To find the minimum point, we must calculate the derivative of the function, set it equal to zero, and solve the equation

$$y' = f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = 0.$$

Now we can determine the signs of the derivative in the intervals between the extrema (minima and maxima). If the sign of the derivative changes from positive to negative, then this is a maximum point. If from negative to positive, then a minimum point.

For example.

Find the minimum points and the minimum value of the function

$$y = (x-2)^2(x+5)-1$$

Let's calculate the derivative of this function.

$$\begin{aligned} y' &= [(x+3)^2(x+5)-1]' = [(x^2+6x+9)(x+5)-1]' = \\ &= [(x^3+5x^2+6x^2+30x+9x+45-1)]' = \\ &= (x^3+11x^2+39x-44)' = 3x^2+22x+39. \end{aligned}$$

Let's set the derivative equal to zero and find the roots of the resulting equation.

$$3x^2+22x+39=0$$

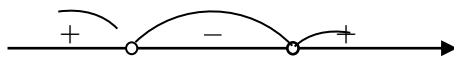
$$x_1 = -3; \quad x_2 = -4\frac{1}{3}$$

Therefore:

$$3x^2+22x+39=3(x+3)(x+4\frac{1}{3})$$

Let's find the intervals where the derivative is positive, i.e., solve the inequality:

$$3(x+3)(x+4\frac{1}{3}) > 0$$



$$-4\frac{1}{3} \quad -3$$

At the point $x = -3$, the derivative of the function changes sign from negative to positive. Therefore, this is a minimum point of the function .

Let's find the minimum of the function . That is, let's find the value of the function at $x = -3$

$$y = (x-2)^2 (x+5) - 1 = (3-2)^2 (3+5) - 1 = 7.$$

We have found the minimum point and the minimum value of the function ` `.

Our student might ask, "What's the point of this?"

Here we can give an example from physics.

Problem. An arrow is launched vertically upward with an initial velocity of 49 m/s. What is the maximum height of the arrow's flight? How many seconds will it take for it to reach this height?

Using the derivative , the problem is solved faster than by the usual method. The height at which the arrow will be depends on time, i.e., $h=f(t)$. This function is given by the formula

$$h = v_0 t + \frac{at^2}{2}, \text{ where } v_0 - \text{initial velocity}$$

(here, $v_0 = 49 \text{ m/s}$); a is the acceleration (here, $a = -9.8 \text{ m/s}^2$).

Let's calculate the derivative of this function.

$$h' = \left(v_0 t + \frac{at^2}{2} \right)' = v_0 + at$$

Substitute the numerical values and set it equal to zero.

$$49 - 9.8 \cdot t = 0$$

$$t = 5 (c)$$

Therefore, the arrow will reach its highest point in 5 seconds. To calculate this height, let's substitute the calculated time into the formula for height:

$$h = v_0 t + \frac{at^2}{2} = 49 \cdot 5 - \frac{9,8 \cdot 5^2}{2} = 122,5 \text{ (m)}.$$

73. Appendix. Basic Level of the Unified State Exam.

We did not prepare specifically for the Unified State Exam. Nevertheless, the lessons we have covered are sufficient to complete the basic-level Unified State Exam tasks in mathematics.

The sample test was taken from the website:

<http://ege.edu.ru/main/demovers/>

In parentheses are links to the page numbers in this book where the corresponding topics are covered.

1. Find the value of the expression

$$\frac{2}{3} + \frac{1}{3} + 2$$

Answer: _____

or

Find the value of the expression

$$\left(\frac{1}{3} + \frac{1}{5}\right) \cdot 6$$

Answer: _____

or

Find the value of the expression

$$1,2 - 6,9 \cdot \frac{1}{2}$$

Answer: _____

2. Find the value of the expression

$$\frac{0,24 \cdot 10^6}{0,6 \cdot 10^4}$$

Answer: _____
or

$$\frac{2^6 \cdot 3^8}{6^5}$$

Answer: _____

3. Personal income tax (PIT) in the Russian Federation is 13% of gross pay. How many rubles does an employee receive after PIT is paid if their gross pay is 20,000 rubles?

Answer: _____
or

25 high school graduates took the physics exam, which is one-third of the total number of graduates. How many graduates from this school did not take the physics exam?

Answer: _____

4. Find m from the equation $F = ma$, if $F = 84$ and $a = 12$.

Answer: _____
or

Find v_0 from the equation $v = v_0 + at$, if $v = 20$, $t = 2$, and $a = 1$.

Answer: _____

or

Find S , given that $S = v_0 t + \frac{at^2}{2}$, and $v_0 = 6$,
 $t = 2$, $a = -2$.

Answer: _____

5. Find $\cos \alpha$, if $\sin \alpha = 0.8$ and $90^\circ < \alpha < 180^\circ$.

Answer: _____

or

Find $\sin 390^\circ$.

Answer: _____

6. A container of yogurt costs 4 rubles and 60 kopecks. What is the maximum number of containers of yogurt that can be bought for 25 rubles?

Answer: _____

or

A kilogram of carrots costs 40 rubles. Oleg bought 2 kilograms of carrots. How much change should he receive from 100 rubles?

Answer: _____

or

It takes 63 rolls of wallpaper to renovate an apartment. What is the minimum number of packs of wallpaper paste you need to buy to renovate the apartment, if 1 pack of paste is enough for 6 rolls?

Answer: _____

7. Find the root of the equation $3^{x-3} = 81$

Answer: _____

or

Find the root of the equation $\log_2(x-3) = 6$

Answer: _____

or

Find the negative root of the equation

$$x^2 - x - 6 = 0.$$

Answer: _____

8. A plot of land for building a health resort is rectangular, with sides measuring 900 m and 400 m. One of the longer sides of the plot runs along the sea, and the other three sides need to be fenced off. Find the length of this fence. Give your answer in meters.

Answer: _____

or

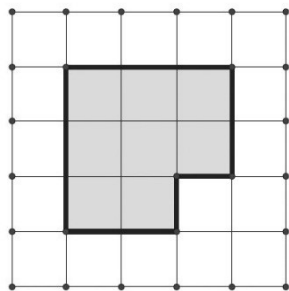
An apartment has two rectangular rooms. The dimensions of the first room are $6\text{ m} \times 3\text{ m}$, and the dimensions of the second room are $5\text{ m} \times 4\text{ m}$. Which of these rooms has a larger area? Write the area of this room in square meters as your answer.

Answer: _____

or

A map of the area is divided into cells. Each cell represents a $10\text{ m} \times 10\text{ m}$ square. Find the area of the section shown on the map. Give your answer in m^2 .

Answer: _____



9. Match the quantities with their possible values: for each item in the first column, select the corresponding item from the second column.

VARIABLES	POSSIBLE VALUES
A) a child's height	1) 32 <i>km</i>
B) thickness of a sheet of paper	2) 30 <i>m</i>
B) Length of a bus route	3) 0.2 <i>mm</i>
D) the height of a residential building	4) 110 <i>cm</i>

In the table, under each letter corresponding to a quantity, indicate the number of its possible value.

Answer:

A	B	C	D

or

Match the quantities with their possible values: for each item in the first column, select the corresponding item from the second column.

TERMS

POSSIBLE
VALUES

A) weight of an adult

1) 8 tons

B) Weight of a truck

2) 5 *g*

B) Weight of a book

3) 65 *kg*

D) the weight of a button on clothing

4) 300 *g*

In the table, list the number corresponding to each possible value under the letter representing the quantity.

Answer:

A	B	C	D

10. Thirty-five athletes are participating in a diving championship: 7 from Russia, 12 from China, 9 from Japan, and 7 from the United States. The order in which the athletes compete is determined by a draw. Find the probability that the athlete competing first will be a representative of Russia.

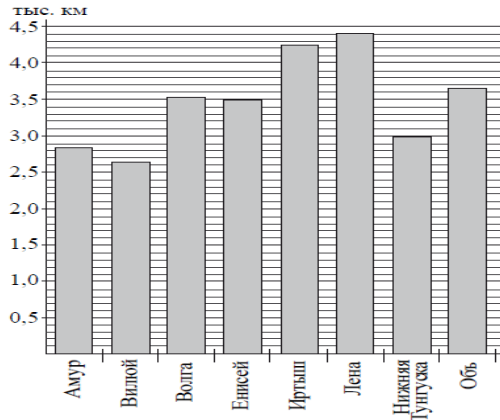
Answer: _____

or

On average, 3 out of every 100 light bulbs delivered to a store are defective. What is the probability that a light bulb randomly selected in the store will be in working order?

Answer: _____

11. The diagram shows data on the lengths of the eight longest rivers in Russia. The Lena River ranks first in length. What is the Amur River's rank in terms of length?



Answer: _____

or

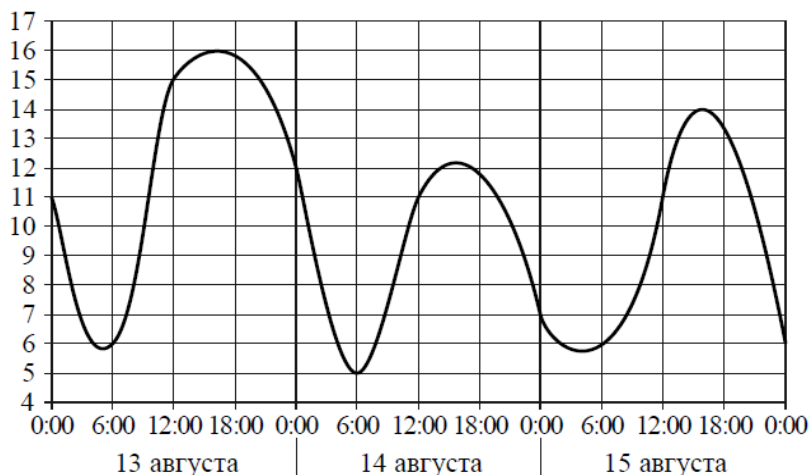
The table shows the distribution of medals at the Winter Olympics in Sochi among the teams that ranked in the top 10 by number of gold medals. How many silver medals does the team that ranked second by number of gold medals have?

Rank	Teams	Medals			
		Gold	Silver	Bronze	Total
1	Russia	13	11	9	33
2	Norway	11	5	10	26
3	Canada	10	10	5	25
4	US	9	7	12	28
5	Netherlands	8	7	9	24
6	Germany	8	6	5	19
7	Switzerland	6	3	2	11
8	Republic of Belarus	5	0	1	6
9	Austria	4	8	5	17
10	France	4	4	7	15

Answer: _____

or

The graph shows the change in air temperature over a three-day period. The x-axis shows the time of day in hours, and the y-axis shows the temperature in degrees. Use the graph to determine the highest air temperature on August 15.



Answer: _____

12. A team of interpreters needs to be assembled to provide services for an international seminar. Information about the candidates is presented in the table.

Inter- preters	Languages	Cost of services (rubles per day)
1	German, Spanish	7,000
2	English, German	6,000
3	English	3,000
4	English, French	6,000
5	French	2000
6	Spanish	4000

Using the table, assemble at least one group in which the translators collectively speak four foreign languages: English, German, French, and Spanish, and the total cost of their services does not exceed 12,000 rubles per day.

In your answer for the selected group, list the translators' numbers without spaces, commas, or other additional characters.

Answer: _____

or

A tourist is choosing excursions. Information about the excursions is presented in the table.

Tours	Sites Visited	Cost (rubles)
1	Fortress, country palace	350
2	Art museum	100
3	Park	150
4	Park, art museum	300
5	Park, fortress	300
6	Country mansion	200

Using the table, put together a tour package so that the tourist visits four sites: the fortress, the country palace, the park, and the art museum, and the total cost of the tours does not exceed 600 rubles.

In your answer for the selected package, list the tour numbers without spaces, commas, or other additional characters.

Answer: _____

or

A construction company plans to buy 70 m^3 of foam blocks from one of three suppliers. Prices and delivery terms are shown in the table. How many rubles does the company need to pay for the cheapest purchase with delivery?

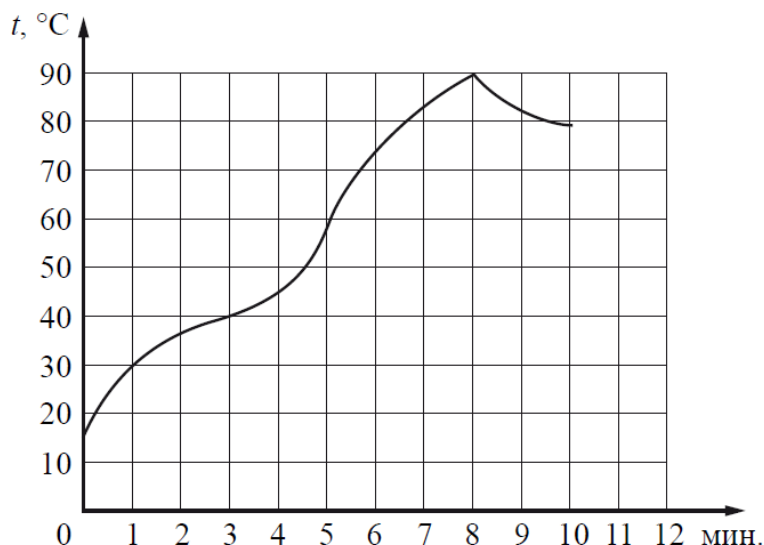
Supplier	Cost of foam blocks (rubles per 1 m^3)	Delivery of delivery (RUB)	Additional delivery terms
A	2,600	10,000	No
B	2,800	8,000	Free shipping on orders over 150,000 rubles
B	2,700	8,000	Free shipping on orders over 200,000 rubles

Answer: _____

13. Water was poured into a cylindrical vessel to a level of 80 *cm*. What level will the water reach if it is poured into another cylindrical vessel whose base radius is 4 times greater than that of the first? Give your answer in *cm*.

Answer: _____

14. The graph shows the warm-up process of a passenger car engine. The x-axis shows the time in minutes elapsed since the engine was started, and the y-axis shows the engine temperature in degrees Celsius.



Using the graph, match each time interval with the characteristic of the engine warm-up process during that interval.

TIME INTERVALS TIME	PROCESS OF THE
A) 0–2 min.	1) the temperature rose most slowly
B) 2–4 min.	2) The temperature dropped
B) 4–6 min.	3) the temperature rose the fastest
D) 8–10 min.	4) The temperature did not exceed 40 °C

In the table, under each letter corresponding to a time interval, indicate the number of the process characteristic.

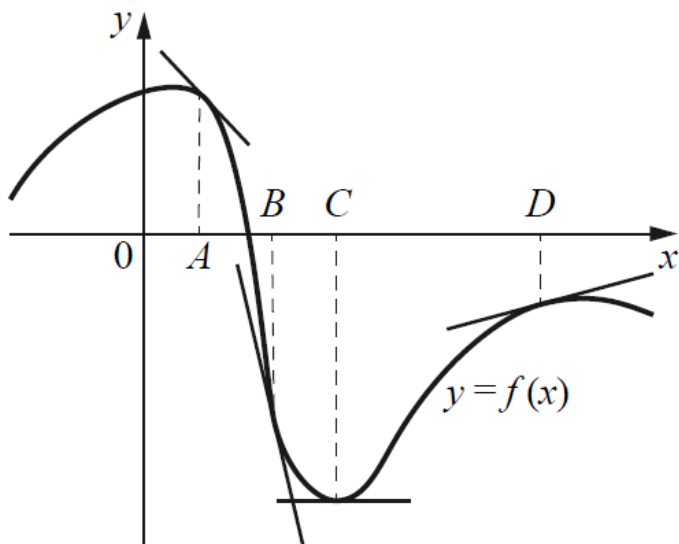
Answer:

A	B	C	D
---	---	---	---

--	--	--	--

or

The figure shows the graph of the derivative of the function $y = f(x)$, with tangents drawn at four points.



The values of the derivative at these points are listed below. Using the graph, match each point with the corresponding value of the derivative.

POINTS

DERIVATIVE VALUES

A

1) 0

B

2) -1

C

3) 0.3

D

4) -5.5

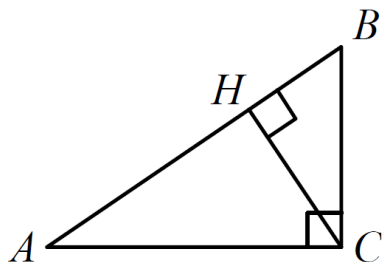
In the table, write the corresponding number under each letter.

Answer:

A	B	C	D
---	---	---	---

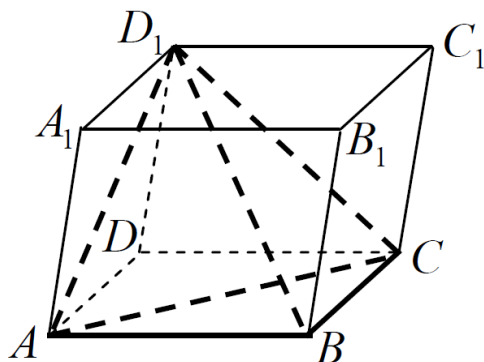


15. In triangle ABC , angle ACB is 90° , $\cos A = 0.8$, $AC = 4$. Segment CH is the altitude of triangle ABC (see fig.). Find the length of segment AH .



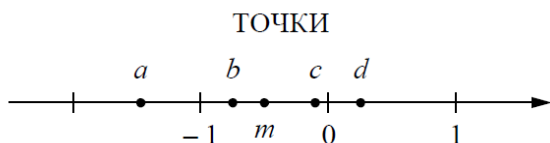
Answer: _____

16. The volume of the rectangular prism $ABCD A_1 B_1 C_1 D_1$ is 15. Find the volume of the pyramid $D_1 ABC$ (see fig.).



Answer: _____

17. The numbers a , b , c , d , and m are marked by points on the number line. Match the given points with the numbers in the right-hand column.



ЧИСЛА

1) $m - \frac{1}{4}$

2) $-\frac{m}{2}$

3) $3m$

4) m^3

In the table, write the corresponding number under each letter.

Answer:

a	b	c	d

or

Each of the four inequalities on the left corresponds to one of the solutions shown on the number line on the right.

Match the inequalities with their solutions.

INEQUALITIES

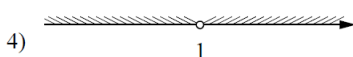
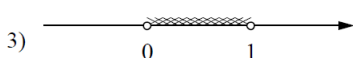
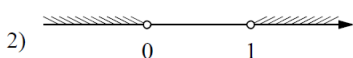
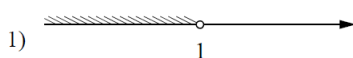
A) $x(1-x) > 0$

B) $1-x=0$

C) $(1-x)^2 > 0$

D) $x(1-x) < 0$

SOLUTIONS



In the table, write the corresponding number under each letter.

Answer:

A	B	C	D

18. In 2013, more boys were born in City Z than girls. The most common name for boys was Andrei, and for girls, Maria. Select the statements that follow from the given data.

Among those born in 2013 in City Z:

1) there are more girls named Maria than girls named Svetlana.

2) there are more boys named Nikolai than boys named Aristarkh.

3) At least one of the boys born was named Andrei.

4) There are more boys named Andrei than girls named Maria.

In your answer, list the numbers of the selected statements without spaces, commas, or other additional characters.

Answer: _____

or

It is known that Vitya is taller than Kolya, Masha is taller than Anya, and Sasha is shorter than both Kolya and Masha. Select the statements that follow from the given data.

1) Vitya is taller than Sasha.

2) Sasha is shorter than Anya.

3) Kolya and Masha are the same height.

4) Vitya is the tallest of them all.

In your answer, list the numbers of the selected statements without spaces, commas, or other additional characters.

Answer: _____

19. Give an example of a three-digit number whose sum of digits is 20, and the sum of the squares of the digits is divisible by 3 but not by 9.

Answer: _____

20. A snail climbs 3 m up a tree during the day and descends 2 m at night. The height of the tree is 10 m . How many days will it take for the snail to reach the top of the tree?

Answer: _____

74. Conclusion

We have covered only a few topics from the school mathematics curriculum. All lessons were conducted according to the following algorithm:

1. *We observed*
2. *Described* what we observed
3. We formulated *a definition*
4. We assigned *a term*
4. *We counted*—we described the number of units in a group, denoting the maximum number of elements in that group with digits and numerals

6. *We measured*—we described size by comparing values to standards.

7. *We calculated*, using relationships between parameters (*formulas*), what is difficult or impossible to measure.

8. *We navigated* using *coordinates* to describe the position of an object in space.

Eight steps...

The first eight steps toward knowledge. And knowledge is an abstraction translated into symbolic form.

Let us note once again that the lessons outlined above are merely intended to assist parents in their conversations with children about schoolwork. For each topic, it is necessary to solve dozens of problems and complete hundreds of exercises to transform knowledge first into understanding, then into skills. This is possible only through systematic work with a professional educator.

Wishing you successful studies and good teachers!

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